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Rearrangements of functions with applications to meteorology and ideal fluid flow

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Contents

1 Introduction 2

2 Rearrangements of scalar valued functions 4
- 2.1 Definition and properties of rearrangements of functions 4
- 2.2 Special rearrangements and rearrangement inequalities 6
- 2.3 Measure-preserving mappings and polar factorisation of scalar valued functions 8
- 2.4 Maximising or minimising functionals with respect to sets of rearrangements 10
- 2.5 Rearrangements of functions on unbounded domains of infinite size 13

3 Application to steady vortices in ideal fluid flow 15
- 3.1 A variational principle for ideal fluid flow 15
- 3.2 Existence of steady vortices 16
- 3.3 A survey of some related ideal fluid flow problems 17

4 Decomposition of weather forecast error using rearrangements of functions 19
- 4.1 A simple formulation of forecast error decomposition 19
- 4.2 Approaches to calculating length of a path in the set of rearrangements . . 22
- 4.3 A characterisation of minimising rearrangements 23
- 4.4 Limitations in the proposed strategy, and an alternative formulation 24

5 Rearrangements of vector valued functions 26
- 5.1 Definition and properties of rearrangements of vector valued functions . . . 26
- 5.2 Monotone rearrangement of vector valued functions 28
- 5.3 Rearrangement inequalities from optimal mass transfer problems 30
- 5.4 Regularity of optimal mass transfers 31
- 5.5 Polar factorisation of vector valued functions 32

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6 Applications of rearrangements of functions to atmospheric and oceanic flow 35
6.1 The semigeostrophic equations 35
6.2 Stable solutions of the semigeostrophic equations as a sequence of minimum energy states 36
6.3 The semigeostrophic equations as a model for frontogenesis 38
6.4 Identification of the trajectory mapping via polar factorisation 39
6.5 Reformulation of the semigeostrophic equations as a coupled Monge-Ampère/transport problem 40
6.6 Weak solutions of the semigeostrophic equations 42

Abstract

A review of the properties of rearrangements of scalar and vector valued functions is presented. Applications of this theory to
(i) steady vortices in ideal fluid flow,
(ii) decomposition of weather forecast error into a combination of displacement error and qualitative features error, and
(iii) a model for large scale motions of the atmosphere and ocean
are described.

1 Introduction

This article studies rearrangements of functions, and considers applications to ideal fluid flow, decomposition of weather forecast error, and a system of equations which models large scale atmospheric and oceanic flow. We begin by giving an intuitive idea of when a function is a rearrangement of another function. Let f be a function defined on a bounded set $\Omega \subset \mathbb{R}^n$, and imagine that Ω is a continuum of infinitesimal particles. Suppose we exchange the particle positions, with each particle retaining its value of f. This yields a function g, which is a *rearrangement* of f. This intuitive notion makes sense whether we attach a scalar or a vector to a particle: the idea of rearranging a function can be applied to both scalar and vector valued functions. Roughly speaking, the formal definition of two functions f and g being rearrangements is that for any given set of values, the set where f takes those values has the same size as the corresponding set for g. (We give precise definitions in §2 and §5.)

For a prescribed function f_0, we can consider the *set of all rearrangements of* f_0. Sets of rearrangements of some given function arise naturally in applications in the following ways:
(i) Suppose we have a quantity q which is conserved following the flow in an ideal (i.e. incompressible, inviscid) fluid. If we consider a fluid particle, it may move as the fluid evolves, but it retains the same value of q, and the particle does not change size (as the flow is incompressible). It follows that at any two instants of time t_1 and t_2, $q(t_1)$ and $q(t_2)$ are rearrangements. (Compare with the intuitive notion of rearrangement above.) In particular, at any time t, $q(t)$ belongs to the set of rearrangements of $q(0)$.
(ii) Suppose we know the values of some quantity on fluid particles at some fixed time t, but do not know the positions of particles, save that they must satisfy conservation of mass. Then the set of possible configurations of the quantity are described by elements of a set of rearrangements.

Without further information, the knowledge that some quantity belongs to a set of rearrangements is of limited utility. However if this is harnessed to a principle that a particular type of flow results from maximising or minimising a functional (of the quantity),

typically an energy, over all possible configurations, we can prove properties of such flows. In §3 we study ideal fluid flow in a bounded planar domain; vorticity is rearrangement preserved in the sense of (i). There is a principle that maximising kinetic energy over a set of rearrangements should yield a steady flow. We review the result that maximising vorticities have corresponding stream functions such that the equation of a steady flow is indeed satisfied. We study the semigeostrophic equations, a standard model for weather front formation, in §6. A vector valued quantity $\mathbf{X}$, from which we can recover the physical quantities velocity, pressure and potential temperature, is predicted on particles at some fixed time t. A constraint on particles is that the flow is incompressible, so possible configurations of $\mathbf{X}$ are given by elements of a set of rearrangements as described in (ii). The *Cullen-Norbury-Purser* principle states that for a solution, the particles are arranged so that geostrophic energy is minimised. We apply this principle at each time t, so we have a constrained minimisation problem, where the constraint changes with t. We review the result that this principle is well posed, and identify such stable solutions with an extra constraint on the system of equations.

This article is organised as follows. Section 2 introduces rearrangements of scalar valued functions. With reference to simple examples, we give a precise definition of two functions being rearrangements, introduce inequalities satisfied by special rearrangements, consider mappings which relate rearrangements to each other, and establish properties of the set of rearrangements. Section 3 presents applications of rearrangement theory to ideal fluid dynamics; in addition to the application mentioned above, we review some related problems. We discuss a new approach to the evaluation of weather forecast error in §4; the error in the forecast of some meteorological quantity is split into a contribution due to displacement, and a part penalising incorrect qualitative features. The decomposition makes use of the results established in §2. Section 5 reviews properties of rearrangements of vector valued functions; as in §2, the various concepts are illustrated by straightforward examples. After introducing the definition of two vector valued functions being rearrangements, and establishing some equivalent properties, we consider inequalities satisfied by a special rearrangement. In general vector valued rearrangement theory is less rich than the scalar valued case. We note the connection between rearrangement inequalities and optimal mass transfer problems. Application of results in §5 to the semigeostrophic equations is the subject of §6. In addition to the identification of stable solutions with an extra constraint that we discussed above, we explain the way in which the semigeostrophic equations model front formation. We review results that have been proved concerning existence of solution for these equations, and indicate the work which remains to achieve a full existence theory. Note that in this article we restrict attention to the semigeostrophic equations with constant rotation. The reader is referred to Cullen and Douglas (1998) for a description of how the methods described in §6 may be applied to the variable rotation case.

This is far from an exhaustive survey of rearrangements of functions and their applications. There is an extensive literature on the use of isoperimetric inequalities, see for example Talenti (1976). Moreover properties of the directional derivative of the increasing rearrangement (see §2 for the definition of the increasing rearrangement of a function) are pertinent to the study of certain differential equations. (See Mossino and Temam 1981.) The reader is referred to Kawohl (1985) for a survey of applications of rearrangements methods to partial differential equations. In this article we focus on those applications studied during the programme *Mathematics of Atmosphere and Ocean Dynamics*, held at the Isaac Newton Institute for Mathematical Sciences in 1996.

2 Rearrangements of scalar valued functions

This section is concerned with rearrangements of scalar valued functions. We introduce the concept via an example in §2.1; we then give a formal definition and note equivalent properties. Applications of rearrangement theory make use of inequalities which hold for special rearrangements: in §2.2 we introduce the increasing and decreasing rearrangements of a function, and discuss inequalities they satisfy. Measure-preserving mappings, which relate rearrangements, are reviewed in §2.3. We recall the result that any real integrable function on a bounded interval can be written as the composition of its increasing rearrangement with a measure-preserving mapping. In applications we often wish to maximise or minimise a functional (which typically represents an energy) over the set of rearrangements of a prescribed function: in §2.4 we establish properties of this set. A sequence of rearrangements of a prescribed function may have a (weak) limit which is not a rearrangement: we define an enlarged set which contains these limits. An extremum of a functional with respect to this set may or may not be a rearrangement: we illustrate with two simple examples. Finally in §2.5 we extend the definition of rearrangement to functions defined on unbounded domains of infinite size.

2.1 Definition and properties of rearrangements of functions

We give a simple example of two scalar functions which are rearrangements of each other. Let $f(x) = x$ for each $x \in [0,1]$, and let g be defined by

$$g(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases} \tag{1}$$

For each $0 < \alpha < 1$,

$$\int_0^1 (f(x)-\alpha)_+ dx = \frac{1}{2}(1-\alpha)^2 = \int_0^1 (g(x)-\alpha)_+ dx \tag{2}$$

where the subscript $+$ denotes the positive part of the function, that is $h_+(x) = \max\{h(x), 0\}$. (2) says that the area of the region which is bounded above by the graph of f and bounded below by the line $y = \alpha$ is the same as the area of the corresponding regions of g. This is illustrated in Figure 1: the total shaded areas are the same. If $\alpha \geq 1$, then $\int_0^1 (f(x)-\alpha)_+ dx = 0 = \int_0^1 (g(x)-\alpha)_+ dx$, and if $\alpha \leq 0$, $\int_0^1 (f(x)-\alpha)_+ dx = -\alpha + \int_0^1 f(x)dx = -\alpha + \int_0^1 g(x)dx = \int_0^1 (g(x)-\alpha)_+ dx$. Combining the above results, $\int_0^1 (f(x)-\alpha)_+ dx = \int_0^1 (g(x)-\alpha)_+ dx$ for each real α.

We will use this property to make a precise definition of when f and g are *rearrangements* of each other. We follow ideas of Eydeland, Spruck and Turkington (1990). Let Ω be a bounded subset of $\mathbb{R}^n$, and let μ be a measure of the size of subsets of Ω. If $n = 1$, an appropriate choice is length, for $n = 2$, area, and for $n = 3$, volume.

Definition. Let $f, g : \Omega \to \mathbb{R}$ be two integrable functions i.e $\int_\Omega |f(x)| d\mu(x) < \infty$, $\int_\Omega |g(x)| d\mu(x) < \infty$. f is a *rearrangement of* g if

$$\int_\Omega (f(x)-\alpha)_+ d\mu(x) = \int_\Omega (g(x)-\alpha)_+ d\mu(x) \tag{3}$$

for every real α.

For most applications, the measure of size we will use is n-dimensional Lebesgue measure, denoted λ_n, which coincides with length, area or volume when $n = 1, 2$ or 3. (Note, however, that there are "pathological" sets whose size we cannot calculate.) In this article we will restrict attention to measures μ where the size of a set $S \subset \Omega$, denoted $\mu(S)$, is

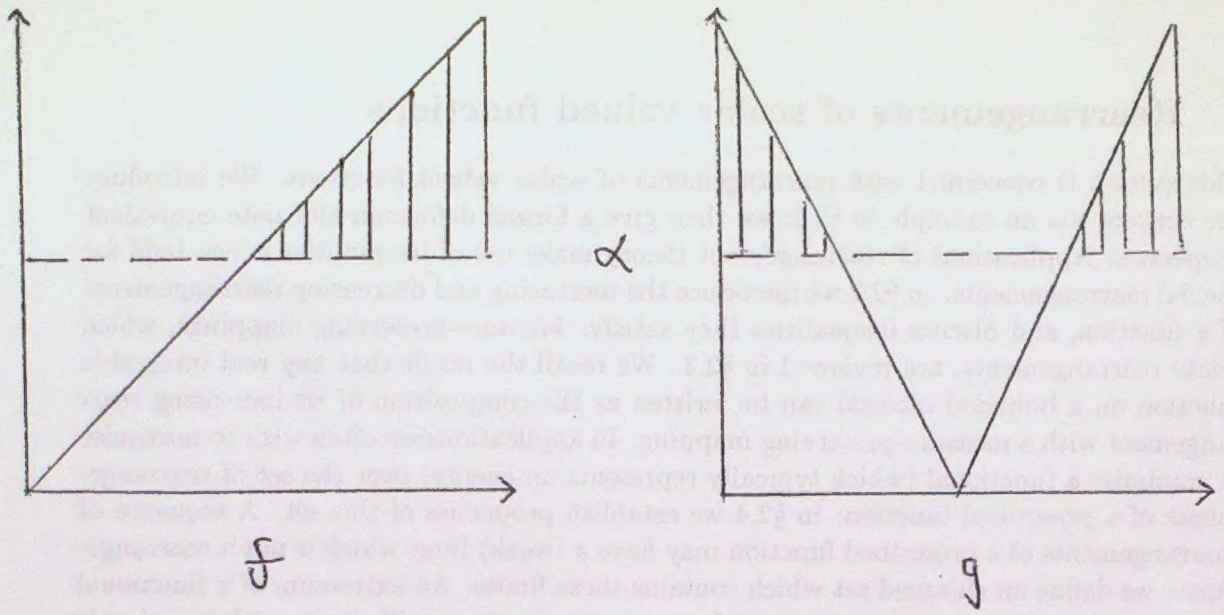


Figure 1: Two rearrangements f and g; the area of the shaded regions is the same for every α

given by $\mu(S) = \int_S h d\lambda_n$, for some non-negative integrable function h. (For a fluid flow problem, such a choice is appropriate if we want the size of the set to be its mass, rather than its volume.) Clearly the choice of h identically equal to 1 recovers n-dimensional Lebesgue measure.

There are other properties which are equivalent to the definition of rearrangement given above.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, and let μ be a measure as above, that is absolutely continuous with respect to n-dimensional Lebesgue measure. Let $f, g : \Omega \to \mathbb{R}$ be integrable functions. Then the following are equivalent.*
(i) f is a rearrangement of g.
(ii) For every real α,

$$\mu(\{x : f(x) \geq \alpha\}) = \mu(\{x : g(x) \geq \alpha\}).$$

(iii) For every (Borel) set $B \subset \mathbb{R}$,

$$\mu(\{x : f(x) \in B\}) = \mu(\{x : g(x) \in B\}). \tag{4}$$

(iv) For each continuous function $F : \mathbb{R} \to \mathbb{R}$,

$$\int_\Omega F(f(x)) d\mu(x) = \int_\Omega F(g(x)) d\mu(x). \tag{5}$$

(Equation (5) is understood in the sense that if one of the integrals is finite, so is the other and they are equal.)

Proof. Follows by the methods of, for example, Douglas (1998). □

Property (ii) is used as the definition of rearrangement in some of the literature. (See, for example, Burton 1987.) For $f, g : [0, 1] \to \mathbb{R}$ we can interpret (ii) as follows. Intersect the graphs of f and g with the line $y = \alpha$; for each function, find the length of that part of the line for which the graph lies on or above the line. f and g are rearrangements if

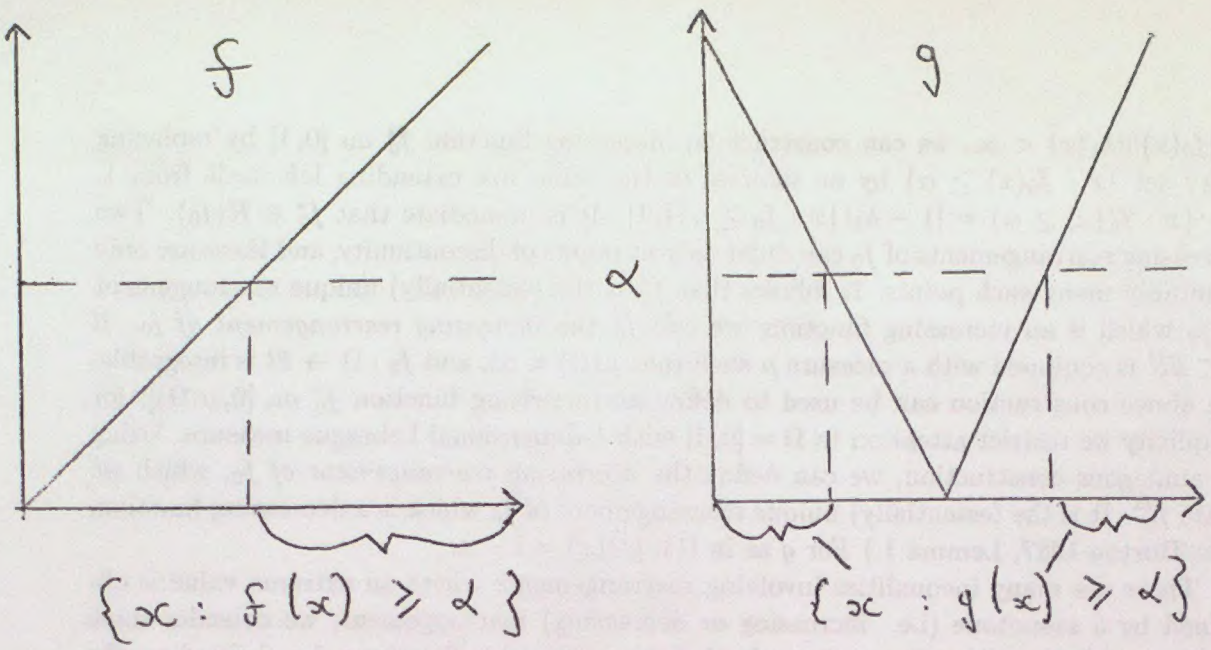


Figure 2: Functions f and g which satisfy Theorem 1 (ii) for every α

the two lengths are equal for every choice of α. For f defined by $f(x) = x$ and g as in (1), this length is 0 when $\alpha \geq 1$, $1 - \alpha$ for $0 \leq \alpha \leq 1$, and 1 for $\alpha < 0$. This is illustrated in Figure 2 for a particular choice of α.

Our intuitive notion of f and g being rearrangements is that for any collection of values, the set where f takes those values has the same size as the corresponding set for g. This is stated precisely in Theorem 1 (iii); the restriction to Borel subsets of $\mathbb{R}$ excludes some "strange" sets, and ensures that we can measure the size of the sets in (4). It is immediate from Theorem 1 (iv) that if $\int_\Omega |f|^p d\mu$ is finite for some $p \geq 1$, and g is a rearrangement of f, then $\int_\Omega |f|^p d\mu = \int_\Omega |g|^p d\mu$. More generally any "reasonable" function F satisfies (5).

We now introduce the concept of the *set of rearrangements* of a given function. A function has more than one rearrangement (unless it is constant). For $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$, the function g given by (1) is a rearrangement of f. A different rearrangement h is defined by $h(x) = 1 - x$. In general the set of all the rearrangements of a given function will have infinitely many elements: as all the members of the set are rearrangements of each other, any one can be used to identify the set. (Rearrangement is an equivalence relation.) We write $\mathcal{R}(f_0)$ to denote the set of rearrangements of a prescribed function f_0.

2.2 Special rearrangements and rearrangement inequalities

In this subsection we identify some special elements of the set of rearrangements of a prescribed function, and review inequalities that they satisfy. Such inequalities are pertinent to the study of maximising or minimising functionals (which often represent an energy in physical applications) with respect to a set of rearrangements; we discuss this in §2.4. Our aim is to introduce results that we will require for applications in §3 and §4, and to review some other important rearrangement inequalities; more on this topic can be found in Alvino, Lions and Trombetti (1989) and Burton (1987).

The first special class of rearrangements we shall consider are monotone rearrangements defined on a (bounded) interval. Let $f(x) = x$ for each $x \in [0,1]$, and let g be as in (1). f is an increasing function, and it can be proved that it is the only increasing function which is a rearrangement of g. More generally, if $f_0 : [0,1] \to \mathbb{R}$ is an integrable function i.e.

$\int_0^1 |f_0(x)| d\lambda_1(x) < \infty$, we can construct an increasing function f_0^* on $[0,1]$ by replacing every set $\{x : f_0(x) \geq \alpha\}$ by an interval of the same size extending leftwards from 1, i.e. $\{x : f_0^*(x) \geq \alpha\} = [1 - \lambda_1(\{x : f_0 \geq \alpha\}), 1]$. It is immediate that $f_0^* \in \mathcal{R}(f_0)$. Two increasing rearrangements of f_0 can differ only at points of discontinuity, and there are only countably many such points. It follows that f_0^* is the (essentially) unique rearrangement of f_0 which is an increasing function; we call f_0^* the *increasing rearrangement of* f_0. If $\Omega \subset \mathbb{R}^d$ is equipped with a measure μ such that $\mu(\Omega) < \infty$, and $f_0 : \Omega \to \mathbb{R}$ is integrable, the above construction can be used to define an increasing function f_0^* on $[0, \mu(\Omega)]$; for simplicity we restrict attention to $\Omega = [0,1]$ with 1-dimensional Lebesgue measure. Using an analogous construction, we can define the *decreasing rearrangement of* f_0, which we write $f_0^{\triangle}$. It is the (essentially) unique rearrangement of f_0 which is a decreasing function. (See Burton 1987, Lemma 1.) For g as in (1), $g^{\triangle}(x) = 1 - x$.

There are many inequalities involving rearrangements where an extreme value is obtained by a monotone (i.e. increasing or decreasing) rearrangement; we consider some fundamental examples. For scalar valued square integrable functions f, g defined on the unit interval, (that is functions satisfying $\int_0^1 f^2 < \infty, \int_0^1 g^2 < \infty$,)

$$\int_0^1 f(x)g(x)d\lambda_1(x) \leq \int_0^1 f^*(x)g^*(x)d\lambda_1(x) \tag{6}$$

where f^*, g^* are the increasing rearrangements of f, g respectively. If we replace the increasing rearrangements by decreasing rearrangements the inequality still holds. Allowing f and g to vary over $\mathcal{R}(f)$ and $\mathcal{R}(g)$ respectively, which pairs of functions maximise $\int_0^1 fg$? From our previous statements it is immediate that the pairs (f^*, g^*), $(f^{\triangle}, g^{\triangle})$ are maximisers; in general there will be others, where the functions are arranged so that the "big values" of f multiply "big values" of g. If f and g have rearrangements $\tilde{f}$ and $\tilde{g}$ such that $\tilde{f} = \phi \circ \tilde{g}$ for some increasing function ϕ, then $\tilde{f}$ and $\tilde{g}$ achieve equality in (6). If we fix g, there is a unique rearrangement $\hat{f} \in \mathcal{R}(f)$ such that equality holds in (6) if and only if there exists an increasing function ϕ such that $\hat{f} = \phi \circ g$. (The reader is referred to Burton 1987 for proofs of these assertions.) In particular, if the sets $\{x : g(x) = \alpha\}$ have zero size for each $\alpha \in \mathbb{R}$,

$$\int_0^1 f(x)g^*(x)d\lambda_1(x) < \int_0^1 f_0^*(x)g^*(x)d\lambda_1(x) \tag{7}$$

for every rearrangement f of f_0 which is not equal to f_0^*. We can always find $\tilde{f} \in \mathcal{R}(f_0)$ which maximises $\int_0^1 fg$ for fixed g; we prove this in the next subsection.

For f and g as above, it follows from (6) and Theorem 1 (iv) that

$$\left\{\int_0^1 |f^*(x) - g^*(x)|^2 d\lambda_1(x)\right\}^{1/2} \leq \left\{\int_0^1 |f(x) - g(x)|^2 d\lambda_1(x)\right\}^{1/2}. \tag{8}$$

We say that the mapping which takes a function to its increasing rearrangement is *non-expansive* on the space of square integrable functions (as it does not increase the L^2-norm). For $1 \leq p < \infty$, if we replace 2 by p and $1/2$ by $1/p$ in (8), then the inequality still holds. (See Crowe, Zweibel and Rosenbloom 1986, Corollary 1.) Noting that the left hand side of the inequality is the L^p norm of $f^* - g^*$, and that the right hand side is the L^p norm of $f - g$, we see that mapping a function to its increasing rearrangement is non-expansive on L^p for $1 \leq p < \infty$. More generally it may be shown (by the methods of Lieb and Loss 1997, Theorem 3.5) that

$$\int_0^1 J(f^*(x) - g^*(x))d\lambda_1(x) \leq \int_0^1 J(f(x) - g(x))d\lambda_1(x)$$

where $J : \mathbb{R} \to \mathbb{R}$ is a non-negative convex function satisfying $J(0) = 0$. (All the above inequalities still hold if we replace increasing rearrangements by decreasing rearrangements.)

Another important class of rearrangements are the (Schwarz) symmetric decreasing rearrangements. For an integrable function f defined on a bounded set Ω in the plane, the *Schwarz-symmetrisation* $f^\triangle$ is formed by replacing every set $\{x : f(x) \geq \alpha\}$ by a disc centred on the origin of the same area. It follows that $f^\triangle$ depends on $|x|$ only, and is a decreasing function of $|x|$. We can extend this concept to three (or indeed n) dimensions: replace discs by balls, and area by volume. For $n \geq 3$, $f^\triangle$ is usually referred to as the *spherically symmetric rearrangement of* f. The analogous inequalities to (6) and (8) extend to spherically symmetric rearrangements, that is

$$\begin{aligned} \int_\Omega f(x)g(x)d\lambda_n(x) &\leq \int_B f^\triangle(x)g^\triangle(x)d\lambda_n(x), \\ \left\{\int_B |f^\triangle(x) - g^\triangle(x)|^2 d\lambda_n(x)\right\}^{1/2} &\leq \left\{\int_\Omega |f(x) - g(x)|^2 d\lambda_n(x)\right\}^{1/2}, \end{aligned}$$

where B is a ball (centre the origin) of the same size as Ω. Moreover, for a non-negative function f defined on a bounded domain in $\mathbb{R}^n$, having square integrable first order partial derivatives, and vanishing on the boundary of Ω, the following inequality is satisfied:

$$\int_B |\nabla f^\triangle(x)|^2 d\lambda_n(x) \leq \int_\Omega |\nabla f(x)|^2 d\lambda_n(x).$$

The set B is as above. (See Brothers and Ziemer 1988 for related inequalities satisfied by the spherically symmetric rearrangement.)

In the next subsection we introduce the concept of a measure-preserving mapping; these maps relate rearrangements to each other. In particular we will see that any integrable function can be written as the composition of its increasing rearrangement with a measure-preserving mapping.

2.3 Measure-preserving mappings and polar factorisation of scalar valued functions

This subsection studies measure-preserving mappings and their relationship with rearrangements. By way of motivation, we return to our statement in the introduction that for an incompressible flow, any Lagrangian conserved quantity is rearrangement preserved. Let $t \to \chi(t, x)$ be the trajectory of the fluid particle which is at x initially. For any given subset U of the fluid at time t, for each particle in U we can find the original position of the particle at time zero; noting that the flow is incompressible, this set of points has the same size as U. It follows that the trajectory mapping is a measure-preserving mapping, the definition of which we give below. Let q be a quantity which is conserved on fluid particles, and let q_0, q_t be its values at times $0, t$ respectively. Now $q_t \circ \chi(t, .) = q_0$, where $\chi(t, .)$ is a measure-preserving mapping (and $\circ$ denotes composition of functions); we see later that this implies q_t and q_0 are rearrangements, which justifies the claim made in the introduction.

Essentially a measure-preserving mapping between two sets U, V is a mapping which satisfies the following property: given any (measurable) set $W \subset V$, the set of points in U which are mapped to W has the same size as W. More precisely:

Definition Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^d$, and let μ, ν be measures (of size) on U, V respectively, with $\mu(U) = \nu(V)$. A *measure-preserving mapping* $s : U \to V$ satisfies $\mu(\{x : s(x) \in W\}) = \nu(W)$ for each (ν-measurable) set $W \subset V$. Halmos (1950) shows that this is equivalent to requiring that for every ν-integrable function f,

$$\int_U f \circ s d\mu = \int_V f d\nu. \tag{9}$$

We consider some elementary examples when $U = V = [0,1]$, and μ and ν are length. It is easily seen that $s_1(x) = x$ for $x \in [0,1]$ is a measure-preserving mapping. However g as defined in (1) is a measure-preserving mapping which is two to one; it follows that measure-preserving mappings need not be one to one. Such mappings need not be smooth: let $s_2 : [0,1] \to [0,1]$ be defined by

$$s_2(x) = \begin{cases} x + 1/2 & \text{if } x \in [0, 1/2], \\ x - 1/2 & \text{if } x \in (1/2, 1), \\ 0 & \text{if } x = 1. \end{cases}$$

Then s_2 is a measure-preserving-mapping from $[0,1]$ to itself that is discontinuous at $x = 1/2$.

If a measure-preserving mapping s is one to one, and maps measurable sets to measurable sets, then the inverse of s exists and is also a measure-preserving mapping. Such an s is called a *measure-preserving transformation*. We show that measure-preserving mappings and transformations preserve rearrangements. Let bounded sets $\Omega \subset I\!R^n$, $\Omega' \subset I\!R^d$ be such that $\mu(\Omega) = \nu(\Omega')$, where ν and μ are measures satisfying the hypotheses of Theorem 1. Let $f, g : \Omega \to I\!R$ be integrable functions satisfying $f \in \mathcal{R}(g)$, and let $s : \Omega' \to \Omega$ be a measure-preserving mapping. Using the notation $h^{-1}(S) = \{x : h(x) \in S\}$, for each (Borel) set $B \subset I\!R$ we have

$$\begin{aligned} \nu((f \circ s)^{-1}(B)) &= \nu(s^{-1} \circ f^{-1}(B)) = \mu(f^{-1}(B)) \\ &= \mu(g^{-1}(B)) = \nu(s^{-1} \circ g^{-1}(B)) = \nu((g \circ s)^{-1}(B)). \end{aligned}$$

Thus $f \circ s \in \mathcal{R}(g \circ s)$. If s is a measure-preserving transformation, then the reverse implication holds. Furthermore, if (Ω', ν) is the same space as (Ω, μ), then $f \circ s \in \mathcal{R}(f)$, and in particular, for a quantity q which remains constant on fluid particles following the motion of an incompressible flow, $q(t) \in \mathcal{R}(q(0))$.

If there exists a measure-preserving transformation τ between two measure spaces (U, μ) and (V, ν), they have the same measure theoretic structure; we call such spaces *isomorphic*. It may be shown that bounded sets $\Omega \subset I\!R^n$ equipped with measures μ satisfying the hypotheses of Theorem 1 are isomorphic to an interval equipped with length (i.e. one-dimensional Lebesgue measure). We can construct the increasing rearrangement of an integrable function $f : \Omega \to I\!R$ by finding $(f \circ \tau)^*$, where $\tau : [0, \mu(\Omega)] \to \Omega$ is a measure-preserving transformation. Moreover we can apply the inequalities of the previous subsection to $f \circ \tau$.

Any other rearrangement may be obtained from the increasing rearrangement by composition with a suitable measure-preserving mapping. Ryff (1970) proved that for any integrable function f defined on $[0,1]$, there exists a measure-preserving mapping $s : [0,1] \to [0,1]$ such that $f = f^* \circ s$. For integrable functions f defined on Ω as above, we can apply this result to $f \circ \tau$, where $\tau : [0, \mu(\Omega)] \to \Omega$ is a measure-preserving transformation. The existence of a measure-preserving mapping $s : \Omega \to [0, \mu(\Omega)]$ satisfying $f = f^* \circ s$ follows easily. The decomposition into the composition of the increasing rearrangement with a measure-preserving mapping is called a *polar factorisation of* f. The methods of Burton and Douglas (1998) demonstrate that the polar factorisation is unique (i.e. there does not exist a measure-preserving mapping t, with $t \neq s$, such that $f = f^* \circ t$,) if and only if f^* is injective. An immediate consequence of the existence of a polar factorisation is the following result (which we stated without proof in the previous subsection):

Proposition 1 *Let Ω, μ be as in Theorem 1. Let $f, g : \Omega \to \mathbb{R}$ be square integrable functions. Then there exists $\hat{f} \in \mathcal{R}(f)$ such that*

$$\int_\Omega f(x)g(x)d\mu(x) = \int_0^{\mu(\Omega)} f^*(x)g^*(x)d\lambda_1(x)$$

i.e. $\sup_{\tilde{f}\in\mathcal{R}(f)} \int_\Omega \tilde{f}g$ is attained by $\hat{f}$.

Proof. From the discussion above, g has a polar factorisation $g = g^* \circ s$ for some measure-preserving mapping $s : \Omega \to [0, \mu(\Omega)]$. Define $\hat{f} = f^* \circ s$. It is immediate that $\hat{f} \in \mathcal{R}(f)$, and

$$\begin{aligned} \int_\Omega \hat{f}(x)g(x)d\mu(x) &= \int_\Omega f^*(s(x))g^*(s(x))d\mu(x) \\ &= \int_0^{\mu(\Omega)} f^*(x)g^*(x)d\lambda_1(x) \qquad (10) \end{aligned}$$

where (10) follows because s is measure-preserving. □

Note that it is not always possible to obtain one rearrangement from another by composing with a measure-preserving mapping. If $f(x) = x$ for $x \in [0, 1]$, and g is as in (2), no measure-preserving mapping $s : [0, 1] \to [0, 1]$ such that $f = g \circ s$ exists.

2.4 Maximising or minimising functionals with respect to sets of rearrangements

We introduced the concept of the set of rearrangements of a prescribed function in §2.1. In this subsection we study properties of this set, in anticipation of later applications where we will maximise or minimise a functional over a set of rearrangements. A conventional approach is to take a maximising sequence which converges to some limit, and demonstrate that the limit is a maximiser. However we will show by example that a sequence of rearrangements of a fixed function (or any subsequence thereof) need not have a limit which is a rearrangement; in fact this sequence has a (weak) limit which is not a rearrangement of the original function. This motivates our study of the closed convex hull of the set of rearrangements, a set which contains all weak limits of sequences of rearrangements. We consider two simple examples of extremising a functional with respect to a set of rearrangements, then with respect to the closed convex hull of that set. Our examples demonstrate that the extreme value over the latter set may or may not be attained by a rearrangement.

For Ω and μ as in Theorem 1, we work with square integrable functions $f_0 : \Omega \to \mathbb{R}$, that is f_0 which satisfy $\int_\Omega f_0^2 d\mu < \infty$. Functions in this space (which we denote $L^2(\Omega, \mu)$) have a finite "energy", but can have arbitrarily small scale oscillations: therefore it is a natural setting for a physical problem. We define a norm on $L^2(\Omega, \mu)$ by $\|f\|_2 = \{\int_\Omega f^2 d\mu\}^{1/2}$. Elements of this space are integrable.

We now recall some standard definitions, which will enable us to say whether a sequence of functions converges to a limit. For more detail see, for example, Friedman (1982, sections 3.2, 4.10, 4.14).

Definitions. A sequence $(f_n) \subset L^2(\Omega, \mu)$ converges strongly to $f \in L^2(\Omega, \mu)$ if $\|f_n - f\|_2 \to 0$ as $n \to \infty$. A set $\mathcal{F} \subset L^2(\Omega, \mu)$ is *strongly closed* if for every sequence (f_n) in $\mathcal{F}$ which converges strongly to a $f \in L^2(\Omega, \mu)$, say, then the limit function f is actually in $\mathcal{F}$. The *closure* of a set $\mathcal{F}$, denoted $\overline{\mathcal{F}}$, is the smallest closed set which contains $\mathcal{F}$.

A set $\mathcal{F} \subset L^2(\Omega, \mu)$ is *strongly compact* if every sequence in $\mathcal{F}$ has a subsequence which converges strongly to an element of $\mathcal{F}$. A strongly compact set is strongly closed.

It is useful to have another notion of convergence and compactness which can "ignore" small scale oscillations, and thus be more realistically applied to physical data. We give an example. For each $n \in \mathbb{N}$, let $f_n \in L^2([0,2\pi],\lambda_1)$ be defined by $f_n(x) = \sin nx$. The sequence (f_n) does not converge in $L^2([0,2\pi],\lambda_1)$, nor does any subsequence. However the f_n approach the constant function with value 0 in an averaged sense, as $\int_0^{2\pi} f_n g \to 0$ as $n \to \infty$ for any $g \in L^2([0,2\pi],\lambda_1)$. We say that f_n *converges weakly* to 0. We give the formal definition.

Definitions. For all positive integers n, let $f_n \in L^2(\Omega,\mu)$, and further suppose that $f \in L^2(\Omega,\mu)$. We say that f_n *converges weakly* to f if $\int_\Omega f_n g \to \int_\Omega fg$ as $n \to \infty$ for every $g \in L^2(\Omega,\mu)$. Strong convergence implies weak convergence, but not vice versa.

A set $\mathcal{F} \subset L^2(\Omega,\mu)$ is *weakly (sequentially) closed* if the weak limit of every weakly convergent sequence in $\mathcal{F}$ belongs to $\mathcal{F}$. Such sets are strongly closed.

A set $\mathcal{F} \subset L^2(\Omega)$ is *weakly (sequentially) compact* if every sequence in $\mathcal{F}$ has a subsequence converging weakly to an element of $\mathcal{F}$. For any sequence in $\mathcal{F}$ which has a weak limit, the weak limit belongs to $\mathcal{F}$. Strong compactness implies weak (sequential) compactness.

When minimising a functional over some set, a key property that the set may possess is convexity. In $\mathbb{R}^n$ a set is convex if the line joining any two points of the set lies in the set. In the context of square integrable functions we have:

Definition. A set $\mathcal{F} \subset L^2(\Omega,\mu)$ is *convex* if for every $f,g \in \mathcal{F}$, and $\lambda \in [0,1]$, then $(1-\lambda)f + \lambda g \in \mathcal{F}$. The *convex hull* of a set $\mathcal{F}$, denoted conv $\mathcal{F}$, is the smallest convex set which contains $\mathcal{F}$.

If a set is closed and convex, then it is weakly closed.

An element f of a convex set $\mathcal{F}$ is an *extreme point* if it does not lie in the interior of any line segment in the set; that is if $f = (1-\lambda)f_1 + \lambda f_2$ for some $f_1, f_2 \in \mathcal{F}$ and $\lambda \in (0,1)$, then $f = f_1 = f_2$. This notion generalises the idea of a vertex of a convex polyhedron.

We show that the set of rearrangements of a non-negative function $f_0 \in L^2(\Omega,\mu)$ is (strongly) closed. Let $(f_n) \subset \mathcal{R}(f_0)$ be a sequence converging to f, say. Then (8) yields that for each $n \in \mathbb{N}$,

$$\|f_0^* - f^*\|_2 = \|f_n^* - f^*\|_2 \leq \|f_n - f\|_2.$$

It follows that $\|f_0^* - f^*\|_2 = 0$, that is $f_0^* = f^*$. Thus $f \in \mathcal{R}(f_0)$.

However $\mathcal{R}(f_0)$ is not strongly compact, weakly compact, nor convex unless f_0 is a constant function. For example, there are sequences of rearrangements which converge weakly to limits which are not rearrangements of the original function (from which it follows that the set is not weakly compact). Let $f_0 : [0,1] \to \mathbb{R}$ be defined by

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0,1/2), \\ 1 & \text{if } x \in [1/2,1]. \end{cases} \tag{11}$$

Define, for $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0 & \text{if } x \in (m/n, (2m+1)/2n], \\ 1 & \text{if } x \in ((2m+1)/2n, (m+1)/n]. \end{cases} \tag{12}$$

where $m = 0, 1, ..., n-1$. The functions f_3 and f_6 are illustrated in Figure 3. For each $n \in \mathbb{N}$, f_n is equal to zero on a set of length $1/2$, and equal to 1 on a set of length $1/2$, therefore $f_n \in \mathcal{R}(f_0)$. However, given any $g \in L^2(0,1)$, it may be shown that $\int_\Omega f_n g d\mu \to 1/2 \int_\Omega g d\mu$ as $n \to \infty$, that is f_n converges weakly to the constant function with value $1/2$, which is not a rearrangement of f_0.

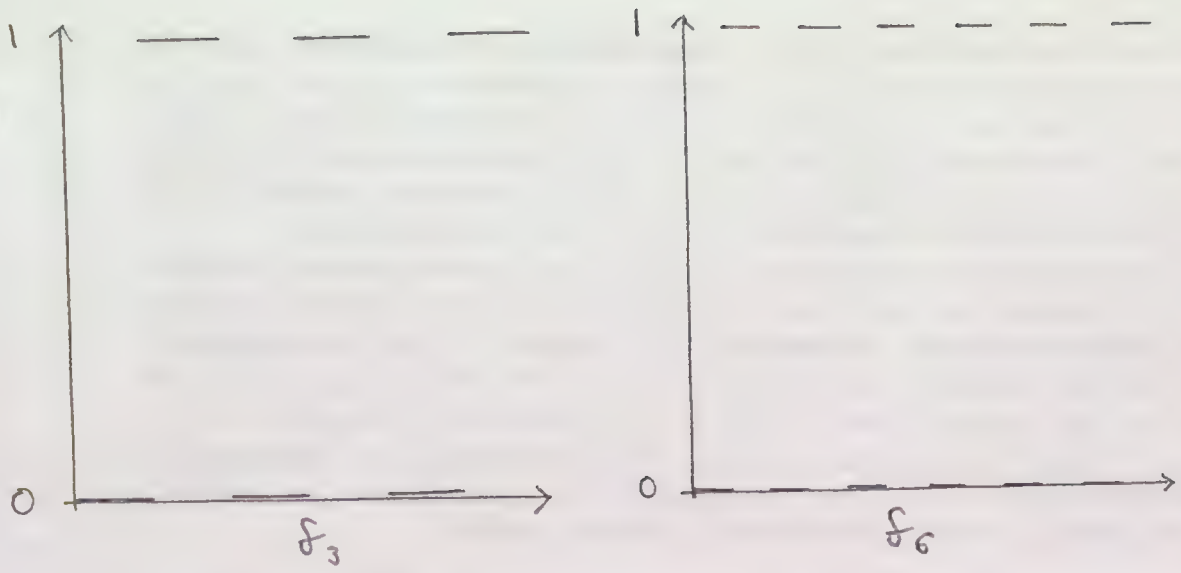


Figure 3: f_3 and f_6 as defined by (12)

This behaviour can make it difficult to maximise or minimise a functional with respect to a set of rearrangements. It is quite possible that there will be a sequence of rearrangements giving progressively less energy, for instance, but the limit function will not be a rearrangement. To compensate for the lack of compactness (and convexity) we hope to use rearrangement inequalities, such as those described in §2.2, to show the existence of maximisers or minimisers. Otherwise we can work with the weak closure of the set of rearrangements of a given function, the smallest weakly closed set that contains the set of rearrangements. It is a weakly compact set: we can extract a weakly convergent subsequence from any extremising sequence, and we may be able to demonstrate that the limit is an extremum. Given a non–negative function $f_0 \in L^2(\Omega)$, the weak closure is equal to the closed convex hull of the set of rearrangements. It is immediate that this set is convex, and it may be shown that its set of extreme points is $\mathcal{R}(f_0)$. Douglas (1994) gave the following characterisation

$$\overline{\text{conv}\, R(f_0)} = \{f \geq 0 : \int_\Omega (f-\alpha)_+ d\mu \leq \int_\Omega (f_0-\alpha)_+ d\mu \ \forall \alpha > 0, \int_\Omega f d\mu = \int_\Omega f_0 d\mu\},$$

where the + subscript denotes taking the positive part of the function. If we choose f_0 as in (11) it can be shown that any integrable function $\varphi : [0,1] \to \mathbb{R}$ satisfying $0 \leq \varphi(x) \leq 1$ for each $x \in [0,1]$, and $\int_0^1 \varphi d\mu = 1/2$, belongs to $\overline{\text{conv}\ R(f_0)}$. This illustrates that $\overline{\text{conv}\ R(f_0)}$ may be a large class of functions, in particular it includes the constant value $1/2$ which is certainly not a rearrangement of f_0. In general, all rearrangements are included, as are functions derived by smoothing a rearrangement, if the smoothing preserves the value of the integral, and does not introduce new extreme values.

We conclude this subsection by considering two simple minimisation problems. Let f_0 be as in (11). Suppose we minimise

$$\text{(i)} \ \int_0^1 (f(x)-x)^2 d\lambda_1(x), \ \text{(ii)} \ \ -\int_0^1 x f(x) d\lambda_1(x)$$

over $f \in \mathcal{R}(f_0)$, then over $f \in \overline{\text{conv}\ \mathcal{R}(f_0)}$. Noting that $\int_0^1 f^2$ is conserved over $f \in \mathcal{R}(f_0)$, the maximiser of $\int_0^1 x f(x)$ over $f \in \mathcal{R}(f_0)$ will be the minimiser of (i). Results in §2.2 yield

that f_0 is the (unique) minimiser, and this gives a value $1/12$ for the integral. However the identity function belongs to $\overline{\text{conv}}\, \mathcal{R}(f_0)$, therefore the minimum of (i) with respect to that set will be zero, that is we obtain a lower value by minimising with respect to the weak closure. The minimum in (ii) with respect to $\mathcal{R}(f_0)$ is again attained by f_0, giving a value $-3/8$. However in this case no lower value may be obtained by minimising with respect to $\overline{\text{conv}}\, \mathcal{R}(f_0)$; indeed it can be shown that f_0 is the unique minimiser.

The difference between these examples follows from the form of the functionals. In the case of (i), $\int f^2$ is conserved under rearrangements of f_0, but is not necessarily conserved when the weak limit is taken. For example, take a sequence (f_n) of rearrangements of f_0 which converges weakly to the identity function id. For each n, $\int_0^1 f_n^2 = 1/2$, but $\int_0^1 id^2 = 1/3$. On the other hand, if (f_n) converges weakly to f, then $\int x f_n \to \int x f$ as $n \to \infty$. In physical applications one has to consider whether it is appropriate to extremise with respect to the set of rearrangements of a given function, or consider a relaxed formulation, extremising with respect to the weak closure.

2.5 Rearrangements of functions on unbounded domains of infinite size

In this article we will only study rearrangements of functions defined on bounded subsets of $\mathbb{R}^n$. However some applications of rearrangement theory are naturally posed on unbounded sets of infinite size. (See for example Douglas 1994.) The properties listed in Theorem 1 are no longer equivalent for functions defined on such domains: we illustrate by an example. Let $f_1, f_2 : [0, \infty) \to \mathbb{R}$ be defined by

$$f_1(x) = \begin{cases} 0 & \text{if } x \in [0,1], \\ 1/x^2 & \text{if } x > 1, \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{if } x \in [0,2], \\ 1/(x-1)^2 & \text{if } x > 2. \end{cases}$$

Now f_1 and f_2 satisfy Theorem 1 (i) and (ii), but not (iii) and (iv). We fix the properties that two functions which are rearrangements satisfy by the following definition.

Definition. Let Ω be an unbounded subset of $\mathbb{R}^n$ of infinite size, that is $\lambda_n(\Omega) = \infty$ where λ_n denotes n-dimensional Lebesgue measure. Two non-negative integrable functions $f, g : \Omega \to \mathbb{R}$ are *rearrangements* if

$$\lambda_n(\{x : f(x) \geq \alpha\}) = \lambda_n(\{x : g(x) \geq \alpha\})$$

for every $\alpha > 0$.

With this definition, f_1 and f_2 defined above are rearrangements. Roughly speaking we are imposing the following condition; for any given set of values not including zero, the set where f takes those values has the same size as the corresponding set for g. However the value zero may be taken on sets of different size. (The concept of a decreasing rearrangement makes sense in this context, but the notion of an increasing rearrangement does not in general.) The key restriction is to functions f such that $\lambda_n(\{x : f(x) \geq \alpha\}) < \infty$ for each $\alpha > 0$. Similar assumptions in the literature are that the functions *vanish at infinity* (see Lieb and Loss 1997, §3.2) or that they are *rearrangeable* (see Simon 1994). Non-negative integrable functions satisfy these conditions.

For non-negative, integrable $f, g : [0, \infty) \to \mathbb{R}$, inequalities (6) and (8) still hold, (replacing the unit interval with the half line,) where f^*, g^* are the decreasing rearrangements of f, g respectively. (See for example Lieb and Loss 1997, Theorems 3.4 and 3.5.) We note an important rearrangement inequality for non-negative integrable functions defined on $\mathbb{R}$. The *symmetric decreasing rearrangement* of a non-negative integrable function $f : \mathbb{R} \to \mathbb{R}$ is constructed by moving each set $\{x : f(x) \geq \alpha\}$ to an interval of the same size, symmetric

about zero. We denote this function $f^{\triangle}$. For non-negative integrable functions f, g and h defined on $\mathbb{R}$, ***Riesz's inequality*** is satisfied:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(x-y)h(y)dxdy \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^{\triangle}(x)g^{\triangle}(x-y)h^{\triangle}(y)dxdy.$$

Riesz's inequality has analogues in higher dimensions. (See Lieb and Loss 1997, Theorem 3.7.) The case when g is the Newtonian potential is of particular interest for kinetic energy minimisation in fluid mechanics.

3 Application to steady vortices in ideal fluid flow

This section discusses applications of rearrangements of functions to ideal fluid flow. For an energy functional written in terms of some Lagrangian conserved quantity, we maximise relative to the set of rearrangements of a prescribed function; maximisers correspond to special flows. We illustrate this variational method by a specific example. Vorticity is preserved following an ideal fluid flow in a bounded planar domain. We show that maximisers of the kinetic energy yield stream functions which satisfy the equation of a steady flow in §3.1. Section 3.2 is concerned with demonstrating that maximisers exist. We review some similar constrained energy maximisation problems in §3.3.

3.1 A variational principle for ideal fluid flow

In this subsection we seek steady flows of an ideal fluid by maximising a functional over a set of rearrangements. Consider an ideal (i.e. incompressible and inviscid) fluid of unit density flowing without body forces in a planar domain Ω bounded by a simple closed curve $\partial\Omega$. The fluid velocity $\mathbf{u}$ and pressure p satisfy the incompressible Euler equations

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} &= -\nabla p, \\ \nabla.\mathbf{u} &= 0. \end{aligned} \tag{13}$$

We impose the boundary condition $\mathbf{u}.\mathbf{n} = 0$ on $\partial\Omega$. The vorticity is given by $\nabla \wedge \mathbf{u}$, and it may be written $\omega\mathbf{k}$ where $\mathbf{k}$ is the unit vector perpendicular to the plane of Ω. Taking the curl of (13), we see that the time derivative following the flow of ω is zero, that is ω is preserved on particles. Moreover the flow is incompressible, therefore (as described in §2.3) for each time t we have $\omega(t) \in \mathcal{R}(\omega(0))$. Noting Theorem 1 (iv), an alternative way to express this fact is that the infinite family of Casimir integrals

$$C_F = \int_\Omega F(\omega) d\lambda_2$$

is conserved (for all t), where F is an arbitrary function (which can be approximated by continuous functions). Therefore we are restricted to a particular *isovortical surface*, or *symplectic leaf*. A *stream function* ψ exists which satisfies

$$\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x)$$

at each time t; it follows that $\omega = -\triangle\psi$. The boundary condition $\mathbf{u}.\mathbf{n} = 0$ implies that ψ is constant on $\partial\Omega$; we can take the constant to be zero. We determine ψ from ω by solving the boundary value problem

$$-\triangle\psi = \omega \text{ in } \Omega, \tag{14}$$

$$\psi = 0 \text{ on } \partial\Omega. \tag{15}$$

For a given square integrable function ω, we can find ψ which satisfies (14) and (15) in the weak sense. (Roughly speaking this means that ψ is zero on $\partial\Omega$ and satisfies

$$\int_\Omega \psi(-\triangle F) d\lambda_2 = \int_\Omega \omega F d\lambda_2$$

for every smooth function F which vanishes on $\partial\Omega$. As in the previous section, λ_2 denotes 2-dimensional Lebesgue measure.) Call this function $K\omega$. This defines a mapping K, which we think of as the inverse of $-\triangle$ (with zero Dirichlet boundary conditions).

We seek solutions of

$$-\Delta\psi = \phi \circ \psi \tag{16}$$

for some stream function ψ, and some function ϕ, where $\circ$ denotes composition. (16) is the equation for the stream function of a steady flow; the steady Euler equation is satisfied with $-p = |\nabla\psi|^2/2 + \Phi\circ\psi$ where $\Phi' = \phi$. We maximise kinetic energy over a family of flows whose vorticities are rearrangements of each other; there is a principle that a maximiser should yield a steady flow. The origins of this idea can be found in the work of Kelvin (1910); the modern formulation using rearrangements of functions is due to Benjamin (1976). (However this was in the context of steady vortex rings in three-dimensional ideal fluid flow - see §3.3 for a discussion of this problem.) The kinetic energy E of the fluid is given by

$$E = \frac{1}{2}\int_\Omega |\mathbf{u}|^2 d\lambda_2 = \frac{1}{2}\int_\Omega |\nabla\psi|^2 d\lambda_2 = \frac{1}{2}\int_\Omega \omega K\omega d\lambda_2.$$

We have the following result (due to Burton 1987) which justifies the above principle.

Theorem 2 *Let $\omega_0 : \Omega \to \mathbb{R}$ be square integrable (i.e. $\omega_0 \in L^2(\Omega)$). Suppose that $\bar{\omega} \in \mathcal{R}(\omega_0)$ is such that $E(\bar{\omega}) \geq E(\omega)$ for each $\omega \in \mathcal{R}(\omega_0)$. Then*

$$-\Delta\bar{\psi} = \phi \circ \bar{\psi} \tag{17}$$

for some increasing function ϕ, where $\bar{\psi} = K\bar{\omega}$.

Proof. (Sketch only.) It is easily seen that K is a linear operator; it may also be shown that

$$\int_\Omega uKvd\lambda_2 = \int_\Omega vKud\lambda_2 \tag{18}$$

for $u, v \in L^2(\Omega)$, and furthermore that $E(\omega) > 0$ if $\omega \neq 0$. Now for $\omega \in \mathcal{R}(\omega_0)$, $\omega \neq \bar{\omega}$,

$$\begin{aligned} E(\bar{\omega}) &\geq E(\omega) \\ &= E(\omega-\bar{\omega}) + \frac{1}{2}\int_\Omega \omega K\bar{\omega}d\lambda_2 + \frac{1}{2}\int_\Omega \bar{\omega}K\omega d\lambda_2 - \frac{1}{2}\int_\Omega \bar{\omega}K\bar{\omega}d\lambda_2 \qquad (19) \\ &= E(\omega-\bar{\omega}) + \int_\Omega (\omega-\bar{\omega})K\bar{\omega}d\lambda_2 + E(\bar{\omega}) \qquad (20) \\ &> \int_\Omega (\omega-\bar{\omega})K\bar{\omega}d\lambda_2 + E(\bar{\omega}). \end{aligned}$$

By way of explanation, we have used linearity of K and (18) to obtain (19) and (20) respectively. Writing $\bar{\psi} = K\bar{\omega}$ we have

$$\int_\Omega \omega\bar{\psi}d\lambda_2 < \int_\Omega \bar{\omega}\bar{\psi}d\lambda_2$$

for every $\omega \in \mathcal{R}(\omega_0)$, $\omega \neq \bar{\omega}$. Burton (1987, Theorem 5) yields that $\bar{\omega} = \phi \circ \bar{\psi}$ for some increasing function ϕ. (This result was discussed in §2.2.) We identify $\bar{\omega}$ with $-\Delta\bar{\psi}$ and obtain (17). □

3.2 Existence of steady vortices

In the previous subsection we saw that a maximiser of the kinetic energy with respect to the set of rearrangements of a given function has a corresponding stream function which satisfies the equation of a steady flow. We demonstrate that such maximisers, and hence such steady flows, exist; the fact that E possesses at least one maximiser relative to $\mathcal{R}(\omega_0)$ is due to Burton (1987, 1989).

Theorem 3 *Let $\omega_0 \in L^2(\Omega)$. Then there exists $\bar{\omega} \in \mathcal{R}(\omega_0)$ such that $E(\bar{\omega}) \geq E(\omega)$ for every $\omega \in \mathcal{R}(\omega_0)$.*

Proof. Firstly we note that for $\omega \in \mathcal{R}(\omega_0)$,

$$E(\omega) \leq \|\omega\|_2 \|K\omega\|_2 \leq \|K\| \, \|\omega\|_2^2 = \|K\| \, \|\omega_0\|_2^2, \tag{21}$$

where $\|.\|_2$ denotes the L^2 norm, and $\|K\|$ is the (finite) operator norm of $K : L^2(\Omega) \to L^2(\Omega)$. By way of explanation, the first inequality in (21) follows by the Cauchy-Schwarz inequality, and the equality by Theorem 1. It follows that $\sup_{\omega \in \mathcal{R}(\omega_0)} E(\omega)$ is finite.

Let (ω_n) be a maximising sequence for E relative to the weak closure of the set of rearrangements of ω_0, which we denote $\overline{\mathcal{R}}(\omega_0)$. (This set, and its properties, were discussed in §2.4.) $\overline{\mathcal{R}}(\omega_0)$ is weakly sequentially compact therefore (ω_n) has a subsequence, which we again denote (ω_n), which converges weakly in $L^2(\Omega)$ to $\hat{\omega}$, say, in $\overline{\mathcal{R}}(\omega_0)$. The linear operator $K : L^2(\Omega) \to L^2(\Omega)$ is compact: consequently $K\omega_n \to K\hat{\omega}$ strongly in $L^2(\Omega)$ as $n \to \infty$. It follows that $E(\omega_n) \to E(\hat{\omega})$ as $n \to \infty$.

Write $K\hat{\omega} = \psi$. Proposition 1 yields the existence of $\bar{\omega} \in \mathcal{R}(\omega_0)$ such that

$$\int_\Omega \bar{\omega}\psi d\lambda_2 = \int_0^{\lambda_2(\Omega)} \omega_0^* \psi^* d\lambda_1.$$

Moreover $\overline{\mathcal{R}}(\omega_0)$ is equal to the closed convex hull of $\mathcal{R}(\omega_0)$, so no larger value can be obtained by maximising $\int_\Omega \omega\psi$ over $\omega \in \overline{\mathcal{R}}(\omega_0)$: in particular,

$$\int_\Omega \bar{\omega}\psi d\lambda_2 \geq \int_\Omega \hat{\omega}\psi d\lambda_2. \tag{22}$$

Repeating a calculation from the proof of Theorem 2, we have

$$E(\bar{\omega}) \geq \int_\Omega (\bar{\omega} - \hat{\omega})\psi d\lambda_2 + E(\hat{\omega}) \geq E(\hat{\omega}),$$

where the second inequality follows from (22). It follows that $E(\bar{\omega}) \geq E(\omega)$ for every $\omega \in \mathcal{R}(\omega_0)$. □

An alternative principle could be proposed; minimise the kinetic energy over a family of vorticities which are rearrangements of each other. If ω_0 is one-signed, then Burton (1989) yields the existence of a unique minimiser $\bar{\omega}$ of E relative to $\mathcal{R}(\omega_0)$ which satisfies $\bar{\omega} = \phi \circ K\bar{\omega}$ for a decreasing function ϕ. However Burton and McLeod (1991) demonstrate that if ω_0 takes both positive and negative values, no minimiser of E relative to $\mathcal{R}(\omega_0)$ exists. (For such ω_0, a unique minimiser $\hat{\omega}$ for E relative to $\overline{\mathcal{R}}(\omega_0)$ does exist; moreover $\hat{\omega} = \phi \circ K\hat{\omega}$ for some decreasing ϕ. Minimisers are characterised in Burton and McLeod,Theorem 2.1.)

3.3 A survey of some related ideal fluid flow problems

Benjamin (1976) proposed a theory of steady vortex rings in an ideal fluid involving maximising a functional over a set of rearrangements of a given function. We describe his original formulation. Let (r, θ, z) denote cylindrical coordinates. Consider an axisymmetric three-dimensional ideal fluid flow having zero velocity in the θ-direction, and whose far-field behaviour is a uniform flow in the z-direction. Subject to suitable regularity assumptions, a *Stokes stream function* Ψ exists which is related to the velocity $\mathbf{u}$ by

$$\mathbf{u} = \left(-\frac{1}{r}\frac{\partial \Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial \Psi}{\partial r} \right)$$

at each time t. The vorticity $\nabla \times \mathbf{u}$ is solely in the θ-direction; we denote its magnitude by ω. Define a differential operator $\mathcal{L}$ by

$$\mathcal{L}(.) \equiv -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(.)}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2(.)}{\partial z^2}.$$

It may be shown that

$$\mathcal{L}\Psi = \frac{\omega}{r}.$$

Moreover $\xi = \omega/r$ is a Lagrangian conserved quantity; noting that the flow is incompressible yields that ξ remains a rearrangement of its initial value. The *impulse* defined by

$$\mathcal{I} = \int_{I\!R^3} \xi r^2$$

is also a Lagrangian conserved quantity. We define $\mathcal{K}$ to be an inverse operator for $\mathcal{L}$ with suitable asymptotic conditions at infinity. (We omit the precise details.) The kinetic energy is given by

$$E = \frac{1}{2}\int_{I\!R^3} |\mathbf{u}|^2 = \frac{1}{2}\int_{I\!R^3} \frac{1}{r^2}|\nabla\psi|^2 = \frac{1}{2}\int_{I\!R^3} \xi\mathcal{K}\xi$$

where ψ is the difference between Ψ and the far-field stream function. Benjamin's proposal was to maximise E over $\xi \in \mathcal{R}(\xi_0)$, where ξ_0 is a prescribed non-negative function. subject to the constraint that $\mathcal{I}$ has some given value. He conjectured that maximisers would yield steady vortex rings. For a maximiser ξ, write $\psi = \mathcal{K}\xi$; one seeks to demonstrate that

$$\mathcal{L}\psi = \phi \circ (\psi - \lambda r^2/2) \tag{23}$$

where λ is the Lagrange multiplier corresponding to the constraint on the impulse, and ϕ is some function. λ represents the far-field velocity. The Stokes stream function has the form $\Psi = \psi - \lambda r^2/2$.

Recent results have justified Benjamin's variational approach to some extent. In view of the symmetry, one works on the half-plane $\Pi = \{(r,z) \in I\!R^2 : r > 0\}$. Burton (1999) has proved the existence of maximisers for a relaxed formulation of the problem, where ξ is constrained to lie in the weak closure of the set of rearrangements of ξ_0, for a wide class of non-negative functions ξ_0. If $\hat{\xi}$ is a maximiser, then an equation corresponding to (23) holds. (Note, however, that the problem is now posed on Π.) The maximisers may be rearrangements of ξ_0; however this is not always the case. (See Burton 1999.)

Benjamin (1976) also proposed a second variational formulation; in this case $\lambda > 0$, which represents the velocity of the vortex ring relative to the fluid velocity at infinity, is prescribed. One maximises $E - \lambda\mathcal{I}$ over the set of rearrangements of a given non-negative function. This problem is studied in, for example, Badiani and Burton (1999).

The strategy of maximising an energy relative to the set of rearrangements of a given function has been successfully applied to other problems. Nycander (1995) demonstrated the existence of a localised, stationary, stable vortex in a background flow of constant shear. The flow was modelled by the incompressible two-dimensional Euler equations. The three-dimensional quasi-geostrophic equations are a more realistic model of geophysical flows; for such a flow in an external shear flow, Burton and Nycander (1999) proved the existence of a vortex having the same properties as in the simpler problem. In this case the constraint set was a set of stratified rearrangements, that is functions obtained by rearranging in the plane at each fixed z, where z is the vertical coordinate. This paper is a good introduction to problems of the type discussed in this section.

4 Decomposition of weather forecast error using rearrangements of functions

In this section we consider an application of rearrangements of functions to the evaluation of weather forecast error: the problem is how to compare the forecast of a physical quantity to the values actually observed. We seek a notion of error which favours what a meteorologist would call a good forecast. In §4.1 we define a forecast error which is composed of a contribution due to differences in qualitative features between the forecast and the actual distribution, and a contribution which measures the error due to displacement. To calculate the latter we need to define what we mean by the length of the shortest path joining two rearrangements: approaches to this are discussed in §4.2. Qualitative features error is evaluated by minimising a function over a set of rearrangements: the minimising rearrangements are characterised in §4.3. In §4.4 we note limitations in the simple formulation outlined in §4.1: we discuss a more sophisticated strategy.

4.1 A simple formulation of forecast error decomposition

There have been many attempts to quantify weather forecast error. In this section we consider a new approach to the problem of evaluating the error of a forecast quantity compared to the actual distribution. Most weather forecast errors are simply displacements of significant weather in space or time, such as a rainband arriving a few hours later than expected. Figure 4 is an example (for real meteorological data) of a forecast which has captured the essential features of the true distribution as measured by the radar data, but has a displacement error. Conventional forecast error scores, such as integrating the square of the difference of the two functions and taking the square root, have limited use as they reward conservative forecasts. For example a forecast which is very good except for misplacing the location of a small intense storm by only a few grid lengths may have a poorer score than one that only forecasts a weak storm in that area. A more informative assessment of the two forecasts is that both have some positioning errors, but the latter has failed to capture the correct qualitative features. Following the pioneering work of Hoffman *et al* (1995), we describe a forecast error which is split into two parts: the error due to displacement, and the error due to differences in qualitative features. This measure of error would favour the former forecast in our example above; moreover the two error values (one corresponding to displacement and one to difference in qualitative features) would be descriptive of how the forecasts had failed.

A simple formulation of forecast error decomposition is as follows. Find a displaced version of the forecast which is a best fit to the actual distribution. Use a conventional forecast error score (such as the L^2 difference described above) to evaluate the error between the displaced forecast and the true distribution: this value represents the error due to difference in qualitative features. Now calculate "the length of the shortest path" which connects the forecast and the displaced forecast (in the space of allowable displacements). This value measures the displacement error. The total error is a weighted sum of the two. Figure 5 illustrates the idea of finding a displaced forecast which most closely matches the true distribution. Essentially we identify similar qualitative features in the forecast and actual distribution, and adjust the forecast so that corresponding features are in the same position.

Consider an important quantity q in weather forecasting which is a function of spatial variables, e.g. rainfall, and suppose we have a distribution predicted by a model, and a distribution derived from observations, which we assume to be the true distribution. Suppose $q : \Omega \to \mathbb{R}^d$, where $\Omega \subset \mathbb{R}^n$ is bounded. We will be interested in the cases $n = 2$

Figure 4: A forecast where the error is predominantly due to displacement. Picture courtesy of U.K. Meteorological Office Forecast Calibration Alignment project.

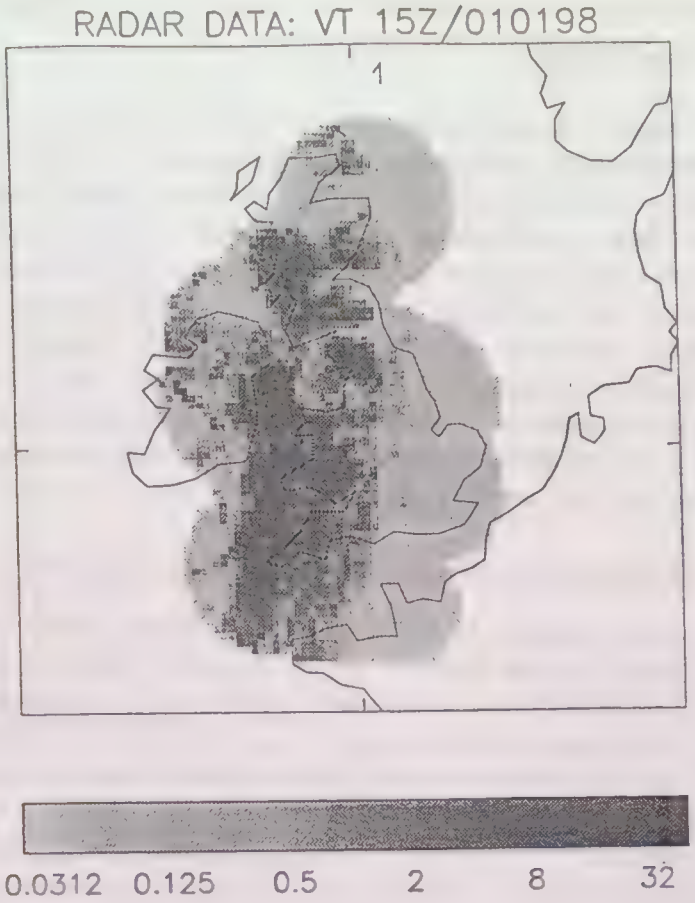
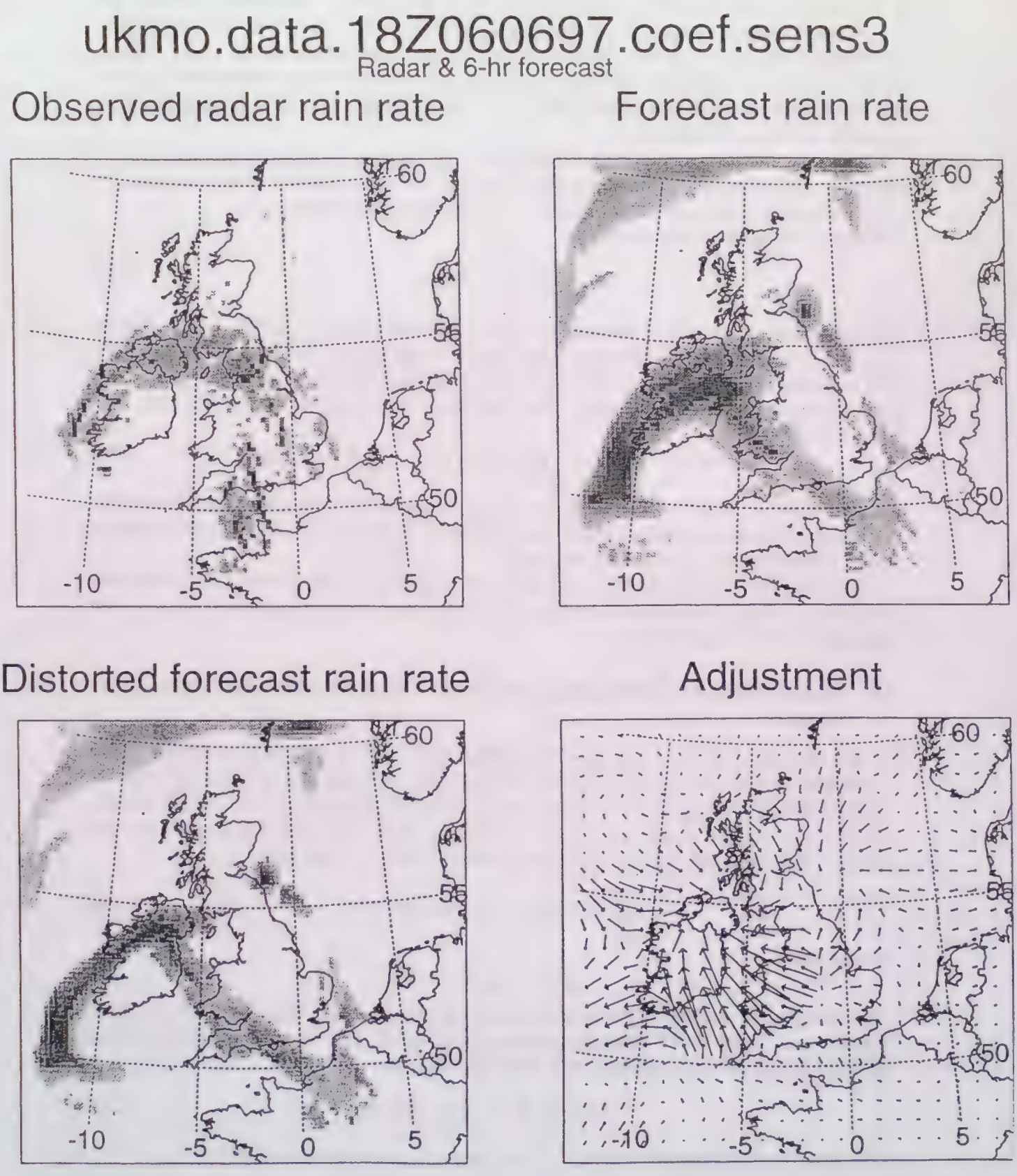


Figure 5: Displacing the forecast to find a best fit to the true distribution. Picture courtesy of U.K. Meteorological Office Forecast Calibration Alignment project.



or 3, and as we have rainfall in mind as the physical quantity, we restrict attention to $d = 1$ and q non-negative. (Note however that the idea of forecast error decomposition still makes sense if $d > 1$: in this case we work with rearrangements of vector valued functions, which are described in §5.) Let q_1 be the observed (true) distribution, q_2 the distribution predicted by the model. We make the physically reasonable assumption that q_1 and q_2 are square integrable.

In a conventional approach the forecast error is $\|q_1 - q_2\|_2 = \{\int_\Omega (q_1(x) - q_2(x))^2 d\lambda_n(x)\}^{1/2}$, where λ_n denotes n-dimensional Lebesgue measure. (For $n = 2$ think of this as area, for $n = 3$, volume.) In our proposed approach we split the error as follows:
(i) Find $\hat{q} \in \mathcal{R}(q_2)$ such that $\hat{q}$ attains

$$\inf_{q \in \mathcal{R}(q_2)} \|q_1 - q\|_2. \tag{24}$$

(ii) Calculate the "length of the shortest path" between q_2 and $\hat{q}$ in $\mathcal{R}(q_2)$. We discuss approaches to defining this quantity, which we write dist $(q_2, \hat{q})$, in the next subsection. (24) represents the error due to difference in qualitative features, whilst dist $(q_2, \hat{q})$ represents the error due to displacement. The total error is a weighted sum of the two, that is

$$TE(q_1, q_2) = (1 - \theta) \inf_{q \in \mathcal{R}(q_2)} \|q_1 - q\|_2 + \theta \inf_{\hat{q} \in \mathcal{M}} \text{dist } (\hat{q}, q_2) \tag{25}$$

where $0 < \theta < 1$, and $\mathcal{M}$ denotes the set of $q \in \mathcal{R}(q_2)$ which attain the infimum in (24). θ is chosen so that one component of error does not dominate the other: to determine a value, one would run a number of test cases.

For our definition of total error to make sense we need to show that (24) is attained by some $\hat{q} \in \mathcal{R}(q_2)$; we demonstrate that this is the case (and establish conditions for the minimiser to be unique) in §4.3.

4.2 Approaches to calculating length of a path in the set of rearrangements

In this subsection we give two possible definitions of the "length of the shortest path" connecting two rearrangements. Firstly we deal with the question of what we mean by a path g which links q_2 and $\hat{q}$ in $\mathcal{R}(q_2)$; we will require that $g : [0,1] \times \Omega \to I\!R$ satisfies $g(0,.) = q_2$, $g(1,.) = \hat{q}$, $g(t,.) \in \mathcal{R}(q_2)$ for each $t \in [0,1]$, and that g is smooth in some sense. We say that the pair (g, v) defines a path if it is a solution of

$$\frac{\partial g}{\partial t} + \nabla.(gv) = 0, \text{ with } \nabla.v = 0, \tag{26}$$

subject to

$$g(0,.) = q_2, g(1,.) = \hat{q}.$$

(We assume the velocity field is sufficiently smooth to apply the transport theory of Diperna and Lions 1989.) Denote the set of paths by G. The distance between q_2 and $\hat{q}$ in $\mathcal{R}(q_2)$ is defined via a Least Action Principle, that is

$$\text{dist } (q_2, \hat{q}) = \inf_{(g,v) \in G} A(g, v), \tag{27}$$

for some Action integral $A = A(g, v)$. Two possible choices are as follows;

$$A_1(g, v) = \int_0^1 \int_\Omega |v(t,x)|^2 dx dt, \tag{28}$$

$$A_2(g, v) = \int_0^1 \int_\Omega g(t,x)|v(t,x)|^2 dx dt. \tag{29}$$

For a given path, (28) penalises the magnitude of the velocity which transports q_2 to $\hat{q}$: thinking of $g(t,.)$ as a density, (29) costs the "kinetic energy" of the transportation. The choice of Action integral will depend on the type of displacement error one wishes to penalise most heavily; this may vary depending on the requirements of the customer using the forecast. Consider a patch of Ω which we move using a specified velocity field: if the values on the patch are high, this will give a bigger value for (29) than if the values are low. In contrast (28) is independent of the values we are transporting. Therefore the choice $A = A_2$ in (27) is more suitable for an application where one is primarily concerned with the position of any heavy rainfall.

If we discard the condition in (26) that the velocity is divergence free, Benamou and Brenier (1998b) show that (27) with $A = A_2$ is a time continuous formulation of an optimal mass transfer problem. (We discuss optimal mass transfer problems in §5.3.) They also describe an augmented Lagrangian numerical technique to compute solutions; one could hope to modify this scheme to calculate (27).

4.3 A characterisation of minimising rearrangements

We demonstrate that the formulation of forecast error decomposition given in §4.1 is well posed: we show that the infimum in (24) is attained, and establish a necessary and sufficient condition for the minimiser to be unique. Next we note a special case when this condition is always satisfied; then we conjecture the extent to which minimisers are determined in the general case. There may be infinitely many minimisers: we give an example. Finally we discuss numerical computation of the relevant quantities.

We have the following result.

Proposition 2 *Let q_1, $q_2 : \Omega \to \mathbb{R}$ be non-negative square integrable functions, where $\Omega \subset \mathbb{R}^n$ is bounded. Then*
(i) there exists $\hat{q} \in \mathcal{R}(q_2)$ such that $\hat{q}$ attains $\inf_{q \in \mathcal{R}(q_2)} \|q_1 - q\|_2$, and
(ii) $\hat{q}$ is the unique minimiser of $\|q_1 - q\|_2$ over $q \in \mathcal{R}(q_2)$ if and only if $\hat{q} = \varphi \circ q_1$ for some increasing function φ.

Proof. (i) Recall from §2.3 that q_1 has a polar factorisation $q_1 = q_1^* \circ s$ for some measure-preserving mapping $s : \Omega \to [0, \lambda_n(\Omega)]$, where q_1^* is the increasing rearrangement of q_1. Let $q \in \mathcal{R}(q_2)$. Now

$$\begin{aligned} \|q_1 - q\|_2 &\geq \|q_1^* - q_2^*\|_2 && (30) \\ &= \|q_1^* \circ s - q_2^* \circ s\|_2 && (31) \\ &= \|q_1 - q_2^* \circ s\|_2. \end{aligned}$$

By way of explanation, (30) follows from Crowe, Zweibel and Rosenbloom (1986, Corollary 1): this inequality was discussed in §2. The alternative characterisation of a measure-preserving mapping given in §2.3 yields (31). It follows that $\hat{q} = q_2^* \circ s$ attains $\inf_{q \in \mathcal{R}(q_2)} \|q_1 - q_2\|_2$.
(ii) From above we see that $\hat{q}$ attains $\inf_{q \in \mathcal{R}(q_2)} \|q_1 - q\|_2$ if and only if $\|q_1 - \hat{q}\|_2 = \|q_1^* - q_2^*\|_2$. Noting from Theorem 1 that the L^2 norm is preserved under rearrangement, it follows that $\hat{q}$ attains $\inf_{q \in \mathcal{R}(q_2)} \|q_1 - q\|_2$ if and only if

$$\int_\Omega q_1 \hat{q} d\lambda_n = \int_0^{\lambda_n(\Omega)} q_1^* q_2^* d\lambda_1. \qquad (32)$$

Burton (1987, Theorems 3 and 5) yields that $\hat{q}$ is the unique element of $\mathcal{R}(q_2)$ where equality holds in (32) if and only if $\hat{q} = \varphi \circ q_1$ for some increasing function φ. (This result was discussed in §2.) □

Remark If q_1 has no level sets of positive measure, that is the sets $L_\alpha = \{x \in \Omega : q_1(x) = \alpha\}$ have zero size for each $\alpha \in \mathbb{R}$, then we can find an increasing function φ such that $\varphi \circ q_1$ is a rearrangement of q_2. Proposition 2 (ii) yields that this is the unique minimiser of $\|q_1 - q\|_2$ over $q \in \mathcal{R}(q_2)$.

Conjecture Minimisers of $\|q_1 - q\|_2$ over $q \in \mathcal{R}(q_2)$ are uniquely determined up to level sets of q_1 with positive measure, and when two minimisers are restricted to any level set of q_1 with positive measure, they are rearrangements on that level set.

We show by example that the minimiser need not be unique. Let $q_1, q_2 : [-1,1]^2 \to \mathbb{R}$ be respectively the actual distribution and forecast of some meteorological quantity, where

$$q_1(x,y) = \begin{cases} 3 & \text{if } x^2+y^2 \le 1/16, \\ 2 & \text{if } 1/16 < x^2+y^2 \le 9/16, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$q_2(x,y) = \begin{cases} 3 & \text{if } x^2+y^2 \le 1/4, \\ 0 & \text{otherwise.} \end{cases}$$

Further define

$$q_3(x,y) = \begin{cases} 3 & \text{if } x^2+y^2 \le 1/16, \text{ or } 6/16 \le x^2+y^2 \le 9/16, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $q_3 \in \mathcal{R}(q_2)$, and moreover

$$\|q_1 - q_2\|_2 = \|q_1 - q_3\|_2 = \inf_{q \in \mathcal{R}(q_2)} \|q_1 - q\|_2.$$

It may be shown that any $\tilde{q} \in \mathcal{R}(q_2)$ which satisfies

$$\{(x,y) : x^2+y^2 \le 1/16\} \subset \{(x,y) : \tilde{q}(x,y) = 3\} \subset \{(x,y) : x^2+y^2 \le 9/16\}$$

is a minimiser. In fact for most choices of piecewise constant functions q_1 and q_2, there will be infinitely many $\tilde{q} \in \mathcal{R}(q_2)$ which attain $\inf_{q \in \mathcal{R}(q_2)} \|q_1 - q\|_2$. We discuss this defect in our formulation in the next subsection.

Finally we note that to calculate (24), the proof of Proposition 2 (i) yields that it suffices to evaluate $\|q_1^* - q_2^*\|_2$. Furthermore if we calculate a polar factorisation of q_1, we have a formula for a rearrangement $\hat{q}$ of q_2 which attains (24). Therefore numerical computation of the relevant quantities consists of finding increasing rearrangements and polar factorisations.

4.4 Limitations in the proposed strategy, and an alternative formulation

We begin by highlighting two limitations in the formulation of forecast error decomposition which we introduced in §4.1: firstly a numerical implementation may require many computations, and secondly errors in qualitative features may be penalised as though they were displacement errors. We describe an alternative formulation which may overcome these difficulties.

As noted in the previous subsection, if we are dealing with piecewise constant data, there may be infinitely many rearrangements of q_2 which attain (24). When we calculate the displacement error, we have to find dist $(q_2, \hat{q})$ for each minimiser $\hat{q}$, taking the least value (if such is attained) as the displacement error. A discretised version of this problem may be computationally expensive.

We illustrate the second difficulty by supposing that we wish to decompose rainfall forecast error over the United Kingdom. It is possible that $\hat{q}$, the rearrangement of the forecast rainfall q_2 which is nearest to the actual distribution q_1, is achieved by rearranging rainfall from a wet area (e.g. Scotland) to a dry area a long distance away (e.g. the

South East of England). This could happen if the forecast overestimated the extent of a storm in the wet area, and failed to predict a small amount of rainfall in the dry area. The displacement error term will be large; however the error is really one of qualitative features. We can lessen the impact of this problem by restricting the region over which we perform the forecast error decomposition to those which are "meteorologically similar".

A possible solution to these difficulties is to minimise the combined quantity of qualitative difference and displacement error over $q \in \mathcal{R}(q_2)$, that is consider the problem

$$\inf_{q \in \mathcal{R}(q_2)} \{(1-\theta)\, \|q_1 - q\|_2 + \theta \text{ dist } (q, q_2)\}, \tag{33}$$

for some $0 < \theta < 1$. If (33) is uniquely attained by $\hat{q} \in \mathcal{R}(q_2)$, then by evaluating $\|q_1 - \hat{q}\|_2$ and dist $(\hat{q}, q_2)$ we recover respectively the difference in qualitative features, and displacement error. (Note, however, that these values need not be the same as the quantities calculated according to the scheme outlined in §4.1.) If we consider the example in §4.3, then q_2 is the unique minimiser of (33), so the above formulation may not suffer from non-uniqueness problems to the extent the formulation in §4.1 does. Benamou and Brenier (1998b) discuss how their numerical scheme (mentioned in §4.2) may be adapted to calculate (33).

5 Rearrangements of vector valued functions

The intuitive idea of when two functions are rearrangements given in the introduction (in terms of exchanging values on particles) is equally applicable to both scalar and vector valued functions. In this section we consider rearrangements of vector valued functions; we seek analogues of the special rearrangements which exist and inequalities which hold in the scalar valued case. We introduce the concept via an example in §5.1, and state some equivalent formulations. Then we consider the generalisation of the increasing rearrangement, the monotone rearrangement, in §5.2: this is the unique rearrangement (of a prescribed vector valued function) equal to the gradient of a convex function. We can establish an inequality satisfied by the monotone rearrangement by considering an appropriate class of optimal mass transfer problems: this is discussed in §5.3. Section 5.4 deals with the regularity of the optimal mappings, in anticipation of an application to a system of equations in §6. Finally we discuss the polar factorisation of a vector valued function, that is writing the function as the composition of its monotone rearrangement with a measure-preserving mapping. We give conditions for when such a decomposition is known to exist, and when it is unique, in §5.5.

5.1 Definition and properties of rearrangements of vector valued functions

We introduce the concept of rearrangement of vector valued functions by an example. Define a function f on the unit square $[0,1] \times [0,1]$, which we write $[0,1]^2$, by

$$f(\mathbf{x} = (x,y)) = \begin{cases} (1,1) & \text{if } 1/2 \le x \le 1, 1/2 \le y \le 1, \\ (0,0) & \text{otherwise,} \end{cases} \tag{34}$$

and another function g by

$$g(\mathbf{x} = (x,y)) = \begin{cases} (0,0) & \text{if } 1/8 < y < 7/8, \\ (1,1) & \text{if } y \le 1/8 \text{ or } y \ge 7/8. \end{cases}$$

f and g are illustrated in Figure 6. Consider the set of points where f takes the value $(1,1)$, that is $\{\mathbf{x} \in [0,1]^2 : f(\mathbf{x}) = (1,1)\}$, and the corresponding set for g, $\{\mathbf{x} \in [0,1]^2 : g(\mathbf{x}) = (1,1)\}$. These sets are equal to $[1/2,1] \times [1/2,1]$ and $[0,1] \times [0,1/8] \bigcup [0,1] \times [7/8,1]$ respectively, and both have area $1/4$. Similarly the sets $\{\mathbf{x} : f(\mathbf{x}) = (0,0)\}$ and $\{\mathbf{x} : g(\mathbf{x}) = (0,0)\}$ have equal area (of $3/4$). As f and g take no other values, it follows that $\{\mathbf{x} : f(\mathbf{x}) \in B\}$ has the same area, or 2–dimensional Lebesgue measure, as $\{\mathbf{x} : g(\mathbf{x}) \in B\}$ for every (Borel) set $B \subset I\!R^2$. We will say that two vector valued functions are rearrangements when this property holds.

More generally we have the following definition.

Definition Let Ω be a bounded set in $I\!R^n$, and let $f, g : \Omega \to I\!R^d$ be integrable functions. Then f and g are *rearrangements* if

$$\mu(\{\mathbf{x} : f(\mathbf{x}) \in B\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \in B\}) \tag{35}$$

for every Borel subset of $I\!R^d$, where μ is the "size" (or measure) of the set. As before an appropriate choice of "size" for $n = 1,2$, or 3 would be length, area or volume respectively. For the purposes of this article we restrict to measures that are of the form $\mu(E) = \int_E f d\lambda_n$ where f is a non-negative integrable function, and λ_n denotes n-dimensional Lebesgue measure. (In the language of measure theory, μ is absolutely continuous with respect to n-dimensional Lebesgue measure. There is no need to be so restrictive for the results that

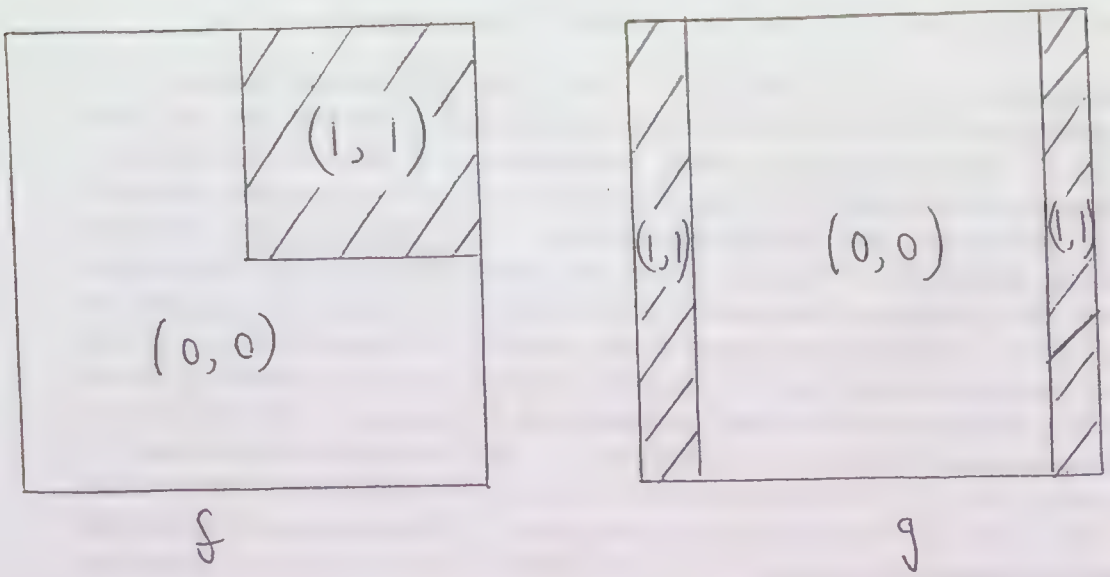


Figure 6: Functions f and g

follow; see Douglas 1998 and Burton and Douglas 1998 for appropriate choices of measure space.) As in the scalar case, the restriction to Borel sets is to ensure that (35) always makes sense.

Definition (35) is identical to (4) for scalar valued rearrangements, except that d will now be greater than 1. Other characterisations of two vector valued functions being rearrangements are given in the following result:

Theorem 4 *Let $\Omega \subset \mathbb{R}^n$ be bounded and let μ be absolutely continuous with respect to n-dimensional Lebesgue measure. Then, for integrable functions $f, g : \Omega \to \mathbb{R}^d$, the following are equivalent.*
(i) f is a rearrangement of g.
(ii) For each $F \in C(\mathbb{R}^d)$ (the set of continuous functions from $\mathbb{R}^d \to \mathbb{R}$)

$$\int_\Omega F(f(x))d\mu(x) = \int_\Omega F(g(x))d\mu(x). \tag{36}$$

((36) is understood in the sense that if one of the integrals is finite, then so is the other and they are equal.)
(iii) For each $c \in \mathbb{R}^d$,

$$\mu(\{x : f(x) \geq c\}) = \mu(\{x : g(x) \geq c\}),$$

where the inequalities are calculated component by component.

Proof. See Douglas (1998, Theorem 2.2). □

Brenier (1991) defined rearrangement of vector valued functions via (ii). (He restricted to a subclass of $C(\mathbb{R}^d)$ so that the integrals in (36) are guaranteed to be finite.) Cullen, Norbury and Purser (1991), following Baigent (1988), used (iii) as a definition; this is a direct extension of the characterisation of scalar valued rearrangements given in Theorem 1 (ii). We can unify these definitions as follows: each may be reduced to the requirement that (36) holds for each $F : \mathbb{R}^d \to \mathbb{R}$ in a class of functions $\mathcal{U}$. The different definitions above correspond to different choices of $\mathcal{U}$. $\mathcal{U}$ must be a sufficiently large class so that for each $n \in \mathbb{N}$ such that $\int_\Omega |f|^n$ is finite, we have $\int_\Omega |f|^n = \int_\Omega |g|^n$, where $|.|$ denotes Euclidean

distance on $\mathbb{R}^d$. (We note that the characterisation of scalar valued rearrangements based on ideas of Eydeland, Spruck and Turkington 1990 can also be generalised to the vector valued case. See Douglas 1998 for details.)

For an integrable function f, we can define the *set of rearrangements of* f, denoted $\mathcal{R}(f)$. We will introduce a special element of this set, the monotone rearrangement of f, in the next subsection, and demonstrate that the set is closed (in L^1). However in general it is neither convex nor compact in the space of integrable functions. Consequently when we minimise a functional with respect to the set of rearrangements of a prescribed integrable function, we cannot assume that every minimising sequence has a subsequence which converges to some rearrangement. Instead we use rearrangement inequalities. (See §6.2.)

A vector valued function $f : \Omega \to \mathbb{R}^d$ may be written in terms of real valued components $f_i : \Omega \to \mathbb{R}$, where $f = (f_1, ..., f_d)$. If f and g are vector valued functions which are rearrangements then the corresponding components of f and g are rearrangements in the scalar valued sense. However it is important to note that the condition that each component of a vector valued function f is a rearrangement (in the scalar sense) of the corresponding component of a vector valued function g, is not sufficient for f to be a rearrangement of g in the vector valued sense. We give an example. Let f be as in the previous example, then

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ 0 & \text{if } \mathbf{x} \notin [1/2,1] \times [1/2,1]. \end{cases}$$

Define

$$g(\mathbf{x}) = \begin{cases} (1,0) & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ (0,1) & \text{if } \mathbf{x} \in [0,1/2] \times [1/2,1], \\ (0,0) & \text{otherwise.} \end{cases}$$

Then $g_1 = f_1$ and

$$g_2(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1/2] \times [1/2,1], \\ 0 & \text{if } \mathbf{x} \notin [0,1/2] \times [1/2,1]. \end{cases}$$

It is easily seen that $f_1 \in \mathcal{R}(g_1)$, and $f_2 \in \mathcal{R}(g_2)$. However, $f \notin \mathcal{R}(g)$, because f takes the value (1,1) on a region of area 1/4, while g never takes this value. Consequently in general we cannot apply scalar valued rearrangement results to the components of vector valued functions and hope to obtain results about vector valued rearrangements. Indeed, the theory of rearrangements of vector valued functions is less rich than that of rearrangements of scalar valued functions. We will see an example of this in the next subsection.

5.2 Monotone rearrangement of vector valued functions

For a real function f defined on an interval, there is an (essentially) unique $f^* \in \mathcal{R}(f)$ which is an increasing function. (See §2.2.) There is an analogous rearrangement for vector valued functions. The concept of increasing does not make sense for vector valued functions, because $\mathbb{R}^d$ is not well–ordered for $d \geq 2$. Instead we note that an increasing scalar valued function is the derivative of a convex function. Replacing derivative with gradient, this is a property which is well defined for vector valued functions, therefore we seek a rearrangement of a vector valued function equal to the gradient of a convex function. Let f be as in (34), and define $f^\# : [0,1]^2 \to \mathbb{R}^2$ by

$$f^\#(\mathbf{x} = (x,y)) = \begin{cases} (1,1) & \text{if } x + y \geq 2 - \frac{1}{\sqrt{2}}, \\ (0,0) & \text{otherwise.} \end{cases} \tag{37}$$

$f^\#$ is easily seen to be a rearrangement of f, and moreover $f^\# = \nabla\Psi$ (at the points where Ψ is differentiable), where $\Psi : [0,1]^2 \to \mathbb{R}$ is a convex function defined by

$$\Psi(\mathbf{x} = (x,y)) = \begin{cases} x + y - \left(2 - \frac{1}{\sqrt{2}}\right) & \text{if } x + y \geq 2 - \frac{1}{\sqrt{2}}, \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that $f^\#$ is the unique rearrangement of f equal to the gradient of a convex function. To generalise this concept to integrable functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^d$, it is clear we require $n = d$. We have the following result.

Theorem 5 *For $n = d$, let Ω and μ be as in Theorem 4. Suppose $u : \Omega \to \mathbb{R}^d$ is an integrable function. Then*
(i) There exists $u^\# \in \mathcal{R}(u)$, such that $u^\# = \nabla\psi$ (at the points of differentiability of ψ), where $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is convex. Moreover $u^\#$ is unique in the sense that if $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is another convex function, and $\nabla\phi \in \mathcal{R}(u)$, then $u^\# = \nabla\phi$ almost everywhere.
(ii) If in addition Ω is open and connected with smooth boundary, and μ is equivalent to d-dimensional Lebesgue measure, then the mapping $u \to u^\#$ is continuous (in $L^1(\Omega, \mu, \mathbb{R}^d)$, the space of integrable functions from $\Omega \to \mathbb{R}^d$).

Proof. (i) follows from the main theorem of McCann (1995). (ii) is part of Brenier (1991, Theorem 1.1). □

We call $u^\#$ the *monotone rearrangement of* u - it is a *cyclically monotone* function, that is it satisfies

$$\sum_{i=1}^{n} u^\#(x_i).(x_i - x_{i-1}) \geq 0$$

for (almost) all finite sequences $x_1, ..., x_n = x_0$ of points in Ω. (For two points, this collapses to the usual definition of monotone.)

For functions f as in Theorem 4, with $n \neq d$, we can still define a monotone rearrangement if we work on a different domain in $\mathbb{R}^d$ which has the same "size" as Ω. Let $\mathcal{B}$ be the open ball in $\mathbb{R}^d$, centre the origin, such that $\mu(\Omega) = \lambda_d(\mathcal{B})$, where λ_d denotes d-dimensional Lebesgue measure. We "move Ω onto $\mathcal{B}$", and work with the resulting function. There exists a measure-preserving transformation (see §2.3) $\tau : (\Omega, \mu) \to (\mathcal{B}, \lambda_d)$. The composition of f with τ satisfies the hypotheses of Theorem 5, and we say that $(f \circ \tau)^\#$ is the monotone rearrangement of f. Using this construction we have a simple proof of the following result.

Proposition 3 *Let Ω, μ, and f be as in Theorem 4. Then $\mathcal{R}(f)$ is closed (in $L^1(\Omega, \mu, \mathbb{R}^d)$).*

Proof. Let $(f_n) \subset \mathcal{R}(f)$, and suppose that $f_n \to g$ in $L^1(\Omega, \mu, \mathbb{R}^d)$ as $n \to \infty$. (Recall that this means that $\int_\Omega |f_n - g| d\mu \to 0$ as $n \to \infty$, where $|.|$ denotes Euclidean distance on $\mathbb{R}^d$.) Let $\mathcal{B}$ be as above, and for each $n \in \mathbb{N}$, let $f_n^\#$ denote the monotone rearrangement of f_n. Then Theorem 5 yields that $f_n^\# \to g^\#$ as $n \to \infty$, and we note that $f_n^\# = f^\#$ for each $n \in \mathbb{N}$. It follows that $g^\# = f^\#$, whence $g \in \mathcal{R}(f)$. □

Recall from §2 that the increasing rearrangement satisfies the following inequalities for two square integrable real valued functions f and g defined on the unit interval:

$$\begin{aligned} \int_0^1 f^*(x) g^*(x) d\lambda_1(x) &\geq \int_0^1 f(x) g(x) d\lambda_1(x), \\ \|f^* - g^*\|_2 &\leq \|f - g\|_2, \end{aligned}$$

where h^* denotes the increasing rearrangement of h, and $\|.\|_2$ the L^2 norm, that is $\|h\|_2 = \{\int_0^1 |h|^2 d\lambda_1\}^{1/2}$. For Ω and μ as in Theorem 5, square integrable functions $u_1, u_2 : \Omega \to \mathbb{R}^d$ do not in general satisfy

$$\begin{aligned} \int_\Omega u_1^\#(x).u_2^\#(x)d\mu(x) &\geq \int_\Omega u_1(x).u_2(x)d\mu(x), \\ \|u_1^\# - u_2^\#\|_2 &\leq \|u_1 - u_2\|_2. \end{aligned} \tag{38}$$

In the next subsection we show that (38) holds in the special case when u_2 is the identity function, and demonstrate that the inequality is strict when $u_1 \neq u_1^\#$. In fact (38) is known to hold when $u_2 : \Omega \to \Omega$ is a measure-preserving mapping. (This result is due to Burton and Douglas 1998, Proposition 2.8.)

5.3 Rearrangement inequalities from optimal mass transfer problems

In this subsection we prove a rearrangement inequality via a reformulation as an optimal mass transfer problem. We begin with a brief review of the latter. The prototype optimal mass transfer problem is the following: given two sets U, V of equal volume, find the optimal volume-preserving mapping between them, where optimality is measured against a non-negative cost function $c = c(x, y)$. One interprets $c(x, y)$ as being the cost per unit mass for transporting material from $x \in U$ to $y \in V$; the optimal map minimises the total cost of redistributing the mass of U through V. Optimal mass transfer problems have a wide range of applications. particularly in economics; in the original problem of Monge (1781), the question was how best to move a pile of soil to an excavation, minimising the work done.

We consider a mathematical formulation of such a problem. Let $U, V \subset \mathbb{R}^d$ be such that $\mu(U) = \nu(V)$, where μ, ν measure "size" on U, V respectively. (More precisely, let μ and ν be Borel measures.) The set of possible strategies for redistributing the mass of U onto V is the set S of measure-preserving mappings between (U, μ) and (V, ν). The problem of finding a strategy which minimises the cost becomes the following; is there a measure-preserving mapping s which attains

$$\inf_{s \in S} \int_U c(x, s(x))d\mu(x), \tag{39}$$

and if so, what is it? This question has been solved for a certain class of cost functions when μ vanishes on sufficiently small sets; the reader is referred to the lucid account of Gangbo and McCann (1996) for details. In what follows we restrict attention to the cost function $c(x, y) = |x - y|^2/2$, where $|.|$ denotes Euclidean distance.

Suppose we are considering the following minimisation problem:

$$\inf_{f \in \mathcal{R}(f_0)} \frac{1}{2} \int_\Omega |f(x) - x|^2 d\mu(x), \tag{40}$$

where Ω and μ are as in Theorem 5, and $f_0 : \Omega \to \mathbb{R}^d$ is a square integrable function. Define a measure ν for (Borel) subsets $B \subset \mathbb{R}^d$ by $\nu(B) = \mu(f_0^{-1}(B))$. Then $\nu(B)$ measures how much of Ω is mapped to the set B. The set of all measure-preserving mappings S from (Ω, μ) to $(\mathbb{R}^d, \nu)$ is exactly $\mathcal{R}(f_0)$, therefore (40) is equivalent to

$$\inf_{f \in S} \frac{1}{2} \int_\Omega |f(x) - x|^2 d\mu(x), \tag{41}$$

which is of the form (39). (To link with notation used elsewhere in the literature, the set S is sometimes referred to as the set of mappings s which push the measure μ forward to the measure ν, which is denoted $s_\# \mu = \nu$.) We have the following result.

Theorem 6 *Let Ω and μ be as in Theorem 5. Suppose $f_0 : \Omega \to \mathbb{R}^d$ is a square integrable function. Then (40) is uniquely attained by $f_0^{\#}$, the monotone rearrangement of f_0, or equivalently*

$$\int_\Omega f_0^{\#}(x).x d\mu(x) > \int_\Omega f(x).x d\mu(x) \tag{42}$$

for each $f \in \mathcal{R}(f_0)\backslash\{f_0^{\#}\}$.

Proof. (41) is uniquely attained by $f_0^{\#}$. (See for example Gangbo and McCann 1996, Theorem 1.2.) The above discussion yields that (40) is uniquely attained by $f_0^{\#}$. This result is easily seen to be equivalent to (42) for every $f \in \mathcal{R}(f_0)\backslash\{f_0^{\#}\}$; write $|f(x) - x|^2$ as $(f(x) - x).(f(x) - x)$, multiply out, and apply Theorem 4. □

In the special case when Ω is open, connected, and has smooth boundary, and μ is equivalent to λ_d, (42) can be deduced from Brenier (1991, Theorem 3.1 and Proposition 3.1 (iii) and (v)). Douglas (1998, Theorem 3.1) showed this result could also be obtained from Brenier (1991, Theorems 1.1, 1.2 and Proposition 2.1).

5.4 Regularity of optimal mass transfers

For two subsets $U, V \subset \mathbb{R}^d$, we considered the problem of optimally transferring the mass of (U, μ) to (V, ν) in the previous subsection, where optimality was measured against the cost function $c(x, y) = |x - y|^2/2$. If there exists some non-negative integrable function g such that $\mu(B) = \int_B g\lambda_d$ for each (Borel) set $B \subset \mathbb{R}^d$ (where λ_d denotes d-dimensional Lebesgue measure), then a unique optimal mapping exists, and it is equal to the gradient of a convex function. In this subsection we are interested in the regularity of the optimal mapping; in §6.6 we study an application of this theory to existence of solutions for a system of partial differential equations. We repeat some of Evans (1997, Section 4); however we quote some additional results for $d = 2$.

Our review will of necessity use technical language; we give a brief guide to the notation used, and refer the reader to Adams (1975) for more details. For $0 < \beta < 1$, $u \in C^\beta(\overline{U})$ if u is continuous on $\overline{U}$, and if there exists $\gamma > 0$ such that for every $x, y \in U$ we have

$$|u(x) - u(y)| \leq \gamma |x - y|^\beta. \tag{43}$$

(This is similar to, but weaker than, a Lipschitz condition.) The Hölder space $C^{1,\beta}(\overline{U})$ consists of $C^1(\overline{U})$ functions (i.e. continuously differentiable up to the boundary of U) for which there exists a $\gamma > 0$ such that (43) holds for every first order derivative (of the function), and every $x, y \in U$. Replacing statements about first order derivatives with second order derivatives, we define $C^{2,\beta}(\overline{U})$.

Denote the optimal mapping between (U, μ) and (V, ν) by $\nabla\Psi$, where Ψ is a convex function. Without further assumptions, little is known about the regularity of Ψ. We will assume that U and V are bounded, connected, open sets in $\mathbb{R}^d$; further suppose that there is a non-negative integrable function h such that $\nu(B) = \int_B h\lambda_d$ for each (Borel) subset B of $\mathbb{R}^d$, and that g and h are bounded above and below, away from zero. If the *target set* V is not convex, then $\nabla\Psi$ can have a singular part: see Caffarelli (1992a) for an example. However if the target set is convex, then Ψ is strictly convex, and belongs to $C^{1,\alpha}_{loc}(U)$ for some $0 < \alpha < 1$. (The subscript *loc* means that the function is $C^{1,\alpha}$ on all compact subsets of U.) If f and g are smoother functions, then so is Ψ. Suppose $g \in C^\beta(U)$ and $h \in C^\beta(V)$ for some $0 < \beta < 1$; then $\Psi \in C^{2,\alpha}_{loc}(U)$ for each $0 < \alpha < \beta$. These interior regularity results are due to Caffarelli (1992a). If U is convex as well, we obtain regularity up to the boundary. For g and h bounded away from 0 and $+\infty$ we have $\Psi \in C^{1,\alpha}(\overline{U})$ for some $0 < \alpha < 1$; if the boundaries of U and V are smooth, and $g \in C^\beta(U)$ and $h \in C^\beta(V)$

for $0 < \beta < 1$, then $\Psi \in C^{2,\alpha}(\overline{U})$ for every $0 < \alpha < \beta$. These assertions were proved (independently) by Urbas (1997) and Caffarelli (1992b, 1996).

For some applications, the hypothesis that both U and V are convex is too restrictive: see §6.3, 6.6. One might hope to make only the assumptions needed for the interior regularity theory, and then demonstrate (partial) regularity up to the boundary. If $d \geq 3$ there is a class of counterexamples due to Pogorelov (1964) which make it difficult to prove further results. However if we restrict attention to $d = 2$, and make the additional assumption that the target set V is strictly convex, then Douglas and McCann (1998) have proved that the $\Psi \in C^1(\overline{U})$ (so we have C^1 regularity up to the boundary). Wolfson (1997) establishes conditions for the existence of smooth area-preserving mappings (which coincide with the optimal mapping) between two domains with smooth boundaries. If the domains satisfy a *pseudo-convexity* condition, that is if

$$\min_{\partial U} \kappa_1 + \min_{\partial V} \kappa_2 > 0, \tag{44}$$

where κ_1 and κ_2 are the minimum curvature values for the curves which comprise ∂U and ∂V respectively, then there is a smooth area-preserving mapping between U and V. Roughly speaking, the pseudo-convexity condition is a joint convexity condition; if U is a disc (which is "as convex as possible"), then (44) may still be satisfied even when V fails to be convex.

5.5 Polar factorisation of vector valued functions

For a real valued integrable function defined on a bounded interval, Ryff (1970) showed that it could be written as the composition of its increasing rearrangement with a measure-preserving mapping. This decomposition is the *polar factorisation* introduced in §2.3. Our purpose in this subsection is to extend this concept to vector valued functions. The monotone rearrangement, introduced in §5.2, plays the role of the increasing rearrangement. The question we address is the following: for a given integrable vector valued function $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$, when does a measure-preserving mapping $s : \Omega \to \Omega$ such that $u = u^\# \circ s$ exist, and when is it unique (i.e. when is it impossible to find a different measure-preserving mapping $t : \Omega \to \Omega$ such that $u = u^\# \circ t$)? The expression $u = u^\# \circ s$ is called a polar factorisation of u; this term was introduced by Brenier (1991).

We begin with a simple example; let f be as (34). The monotone rearrangement of f, $f^\#$, is defined by (37). We seek an area-preserving mapping which maps the set where f takes the value $(1,1)$ to the set where $f^\#$ does, and similarly for $(0,0)$. Denote the set where f takes the value $(1,1)$ by A: the corresponding set for $f^\#$ by $A^\#$. Split the set $A \backslash A^\#$ into two triangles of equal size by bisecting with the line $x = y$; we call the triangle above the line B_1, the one below B_2. Denote the part of $A^\# \backslash A$ above the line $x = y$ by $B_1^\#$, the remainder by $B_2^\#$. Figure 7 illustrates this notation. We can map B_1 onto $B_1^\#$ by a rotation and a translation - we write τ_1 for this map, and τ_2 for the analogous map which moves B_2 onto $B_2^\#$. Define $s : [0,1]^2 \to [0,1]^2$ by

$$s(\mathbf{x}) = \begin{cases} \tau_1(\mathbf{x}) & \text{if } \mathbf{x} \in B_1, \\ \tau_2(\mathbf{x}) & \text{if } \mathbf{x} \in B_2, \\ \tau_1^{-1}(\mathbf{x}) & \text{if } \mathbf{x} \in B_1^\#, \\ \tau_2^{-1}(\mathbf{x}) & \text{if } \mathbf{x} \in B_2^\#, \\ \mathbf{x} & \text{otherwise.} \end{cases}$$

Now s is an area-preserving mapping which satisfies $f = f^\# \circ s$: thus f has a polar factorisation. (It is easily seen that it is not unique.)

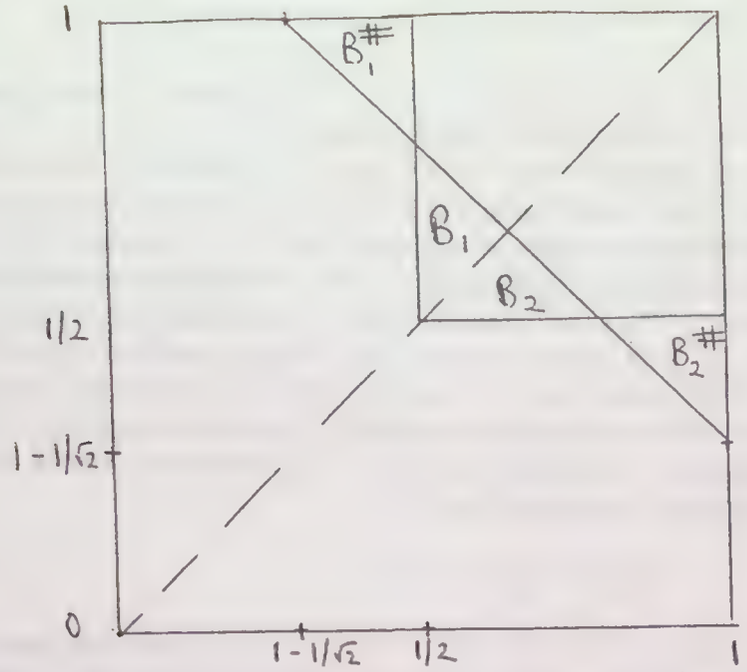


Figure 7: Construction of a polar factorisation

To prove a general result on the existence and uniqueness of polar factorisations, we require some definitions and notation.

Definitions. Let $\mu = \lambda_d$, and let Ω and u be as in Theorem 5. u is *nondegenerate* if u does not map a set of positive size to a set of zero size, or more precisely if $\mu(\{x : u(x) \in E\}) = 0$ for every set $E \subset \mathbb{R}^d$ of zero Lebesgue measure. Otherwise u is *degenerate*.

The function f defined by (34) is degenerate: it maps a set of area $1/4$ to the point $(1,1)$, which has zero area.

u is *countably degenerate* if we can make u nondegenerate by removing countably many level sets. (Level sets of a function are sets of the form $\{x : u(x) = c\}$.)

If we remove the level sets corresponding to $(1,1)$ and $(0,0)$, the function f defined by (34) is nondegenerate: therefore f is countably degenerate.

A function is *almost injective* if it is an injective function except possibly on a set of measure zero. If u is nondegenerate, $u^\#$ is almost injective (see Burton and Douglas 1998, Lemma 2.4).

We have the following result on the existence and uniqueness of a polar factorisation (due to Burton and Douglas 1998).

Theorem 7 *Let $\mu = \lambda_d$, and let Ω and u be as in Theorem 5. Then*
(i) If u is countably degenerate, then u has a polar factorisation $u = u^\# \circ s$, where $s : \Omega \to \Omega$ is a measure-preserving mapping.
(ii) If $u^\#$ is almost injective, then the polar factorisation exists and is unique.

From our earlier remarks, (ii) applies to nondegenerate functions. (The fact that a nondegenerate function has a unique polar factorisation was first established by Brenier 1991, Theorem 1.2.) The idea of the proof of (ii) is as follows. Suppose $u^\#$ is injective - we show that $(u^\#)^{-1} \circ u$ is a measure-preserving mapping. To show uniqueness, suppose $u = u^\# \circ s = u^\# \circ t$ for two measure-preserving mappings $s, t : \Omega \to \Omega$; then injectivity of $u^\#$ implies that s and t must be equal (except possibly on a set of measure zero). For (i), the essence of the proof is to map the level sets of positive measure of u to the corresponding level sets of $u^\#$, and apply (ii) to the nondegenerate function which remains after these sets have been removed.

Can we extend this result? It is easy to see that the polar factorisation cannot be unique if u has a level set of positive measure. $u^\#$ has a corresponding level set of positive measure, and we can choose a non-trivial measure-preserving mapping $\tau : \Omega \to \Omega$ which leaves points other than those in the level set fixed. If $u = u^\# \circ s$ is a polar factorisation of u, then so is $u = u^\# \circ (\tau \circ s)$: s is not equal to $\tau \circ s$. If $d \geq 2$, even when a function fails to have any level sets of positive measure, the polar factorisation, if it exists, may not be unique. (See Burton and Douglas 1998 for an example.) Burton and Douglas conjectured that the polar factorisation always exists, and is unique if and only if the monotone rearrangement is almost injective.

Finally we address the following problem: given a square integrable vector valued function $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$, what is the projection of u onto the set of measure-preserving mappings from Ω to Ω? This question is linked to polar factorisation by the following proposition.

Proposition 4 *Let Ω and μ be as in Theorem 7, and suppose $u : \Omega \to \mathbb{R}^d$ is square integrable. Then*
(i) Measure-preserving mappings $s : \Omega \to \Omega$ which satisfy $u = u^\# \circ s$ maximise $\int_\Omega u(x).s(x)d\mu(x)$ over the set S of measure-preserving mappings from Ω to Ω.
(ii) Suppose that u has a polar factorisation. Then every $s \in S$ which maximises $\int_\Omega u(x).s(x)d\mu(x)$ satisfies $u = u^\# \circ s$.

Proof. (i) Burton and Douglas (1998, Proposition 2.8). (ii) Douglas and McCann (1999). □

We say that a measure-preserving mapping $s : \Omega \to \Omega$ belongs to the projection of u onto S if it is a closest element of S to u where distance is measured by the L^2-norm i.e. it attains

$$\inf_{s \in S} \|u - s\|_2^2 = \inf_{s \in S} \int_\Omega |u(x) - s(x)|^2 d\mu(x).$$

In general there will be many "closest" elements of S to u, therefore the projection will be multi valued. It is easily shown that minimising $\|u - s\|_2^2$ over $s \in S$ is equivalent to maximising $\int_\Omega u(x).s(x)d\mu(x)$ over $s \in S$. If u has a polar factorisation, then Proposition 4 says that s is in the projection of u onto S if and only if s arises from a polar factorisation of u. It follows that if $u^\#$ is almost injective, then the projection of u onto S will be unique. In particular the projection of a nondegenerate function u onto S is single valued. (This result was first proved by Brenier 1991.)

6 Applications of rearrangements of functions to atmospheric and oceanic flow

This section studies the semigeostrophic equations. a model for slowly varying flows constrained by rotation and stratification. They have, in particular, been used to study front formation in meteorology. After stating the equations in §6.1, we show how stable solutions can be interpreted as a sequence of minimum geostrophic energy states in §6.2; the energy minimisation is carried out over a set of rearrangements. At each fixed time t, the minimising rearrangement is the gradient of a convex function; as we discuss in §6.3, tracking singularities of this potential is thought of as weather fronts forming and evolving. In the special case when the minimising rearrangement is nondegenerate (see §5.5), we show in §6.4 that we can find the mapping which relates the Lagrangian and Eulerian variables. Retaining the nondegeneracy assumption, in §6.5 we exploit the so-called *duality structure* of the semigeostrophic equations to reformulate the system as a coupled Monge-Ampère/ transport problem. We discuss existence and uniqueness of solution for this coupled system with reference to relevant results in the literature in §6.6.

6.1 The semigeostrophic equations

Weather systems and equivalent large scale flows in the ocean can be characterised as slowly varying flows constrained by rotation and stratification. A standard model for such flows are the three-dimensional Boussinesq equations of semigeostrophic theory on an f plane. The derivation and validity of the semigeostrophic equations are discussed in greater detail elsewhere in this volume; we note that the system is valid when the timescale for a change in velocity following a fluid particle is much greater than f^{-1}. (See Shutts and Cullen 1987.) In particular this is true in the case of front formation in meteorology, where the flow is largely parallel to the front. We state the equations in the form used by Hoskins (1975).

$$\frac{Du_g}{Dt} - fv_{ag} = 0, \frac{Dv_g}{Dt} + fu_{ag} = 0, \tag{45}$$

$$\frac{D\theta}{Dt} = 0, \tag{46}$$

$$\nabla.\mathbf{u} = 0, \tag{47}$$

$$\nabla\phi = \left(fv_g, -fu_g, \frac{g\theta}{\theta_0}\right) \tag{48}$$

where

$$\begin{aligned} \mathbf{u} &\equiv (u,v,w) \equiv \mathbf{u}_g + \mathbf{u}_{ag}, \\ \mathbf{u}_g &\equiv (u_g, v_g, 0), \\ \frac{D}{Dt} &\equiv \frac{\partial}{\partial t} + \mathbf{u}.\nabla \end{aligned}$$

f is the Coriolis parameter, assumed constant (so we are considering constant rotation), g is the acceleration due to gravity, θ_0 is a reference value of the potential temperature θ, and ϕ is a pressure variable. Subscripts g and ag denote geostrophic and ageostrophic velocity (or wind) components respectively, where the geostrophic velocity is defined to be the horizontal component of velocity in balance with the pressure gradient. This definition is included in equation (48), as is the statement of hydrostatic balance.

We solve the equations (for the velocities $\mathbf{u}$, $\mathbf{u}_g$, potential temperature θ, and pressure variable ϕ) in an open, bounded, connected set $\Omega \subset I\!R^3$ which has smooth boundary, with normal velocity $\mathbf{u}.\mathbf{n} = 0$ on $\partial\Omega$. (In the meteorological literature the equations are sometimes posed on $\overline{\Omega}$.) For $\mathbf{x} = (x, y, z) \in \Omega$, by making the substitution

$$\mathbf{X} \equiv (X, Y, Z) \equiv (x + v_g/f, y - u_g/f, (g/f^2\theta_0)\theta) \tag{49}$$

it is shown in Purser and Cullen (1987) that we may replace (45) and (46) by

$$\frac{D\mathbf{X}}{Dt} = \mathbf{u_g}.$$

It is immediate from (48) and (49) that

$$\mathbf{X} = \nabla P \text{ where } P = \left(\frac{\phi}{f^2} + \frac{1}{2}(x^2 + y^2)\right). \tag{50}$$

(49) is known as the *geostrophic transformation* and is due to Hoskins (1975).

We think of $\mathbf{X}$ as a function of the physical space co-ordinates $\mathbf{x}$. Rewriting in terms of $\mathbf{X}$ and $\mathbf{x}$, we have

$$\frac{D\mathbf{X}}{Dt} = fJ(\mathbf{X} - \mathbf{x}) \tag{51}$$

where

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We now have a system of equations (50), (51) and (47) for unknowns $\mathbf{X}$, $\mathbf{u}$ and ϕ which we solve on $[0, t_1) \times \Omega$, for some $t_1 > 0$. We seek solutions in a restricted class - solutions that are stable in a sense we make precise in the next subsection.
We introduce the Lagrangian coordinates

$$\tilde{\mathbf{X}}(t, \mathbf{x}) = \mathbf{X}(t, \chi(t, \mathbf{x}))$$

where

$$t \to \chi(t, \mathbf{x})$$

is the trajectory of the fluid particle which is at $\mathbf{x}$ initially. The Lagrangian form of (50), (51) and (47) is

$$\tilde{\mathbf{X}}(t, \mathbf{x}) = \nabla P(t, \chi(t, \mathbf{x})), \tag{52}$$

$$\frac{\partial \tilde{\mathbf{X}}}{\partial t} = f(J(\tilde{\mathbf{X}} - \chi)), \tag{53}$$

$$\det D\chi(t, \mathbf{x}) = 1, \tag{54}$$

where $D\chi$ denotes the Jacobian matrix of the mapping χ.

6.2 Stable solutions of the semigeostrophic equations as a sequence of minimum energy states

Cullen, Norbury and Purser (1991) sought solutions of (50), (51) and (47) which can be interpreted as a sequence of constrained minimum energy states, where the constraints evolve with time. They considered solutions obtained by the following procedure. At each fixed time t, predict $\mathbf{X}$ on particles using (51). (In the Lagrangian variables defined above, we predict $\tilde{\mathbf{X}}(t, .)$ but not $\chi(t, .)$.) Now apply the *Cullen-Norbury-Purser* principle which

states that for a solution, the particles are arranged to minimise geostrophic energy. We claim that this determines $\mathbf{X}(t,.)$, (that is a minimiser exists and is unique,) and that $\mathbf{X}(t,.) = \nabla\psi(t,.)$ for some convex function $\psi(t,.)$.

For a solution obtained by the Cullen-Norbury-Purser methods we identify $P(t,.)$ with $\psi(t,.)$ at each time t. Therefore we have a solution of (50), (51) and (47) with $P(t,.)$ convex at each time t. Conversely, if we apply the Cullen-Norbury-Purser methods to a solution of (50), (51) and (47) with $P(t,.)$ convex at each time t, we obtain the given solution. (Recall that the monotone rearrangement is essentially unique.) It follows that seeking solutions that can be interpreted as a sequence of (constrained) minimum energy states is equivalent to solving (50), (51) and (47) with the extra constraint that $P(t,.)$ is convex at each time t. As convexity corresponds to a minimum energy principle, we expect such solutions to be stable (in some sense). In Shutts and Cullen (1987) it is shown that they are stable with respect to small displacements in a frozen pressure field.

We justify our earlier claim that the Cullen-Norbury-Purser principle is well posed i.e. demonstrate that an energy minimiser exists and is unique. The geostrophic energy E is defined as

$$\begin{aligned} E &= \int_\Omega \frac{1}{2}u_g^2 + \frac{1}{2}v_g^2 + \frac{g\theta z}{\theta_0} d\lambda_3(\mathbf{x}) \\ &= f^2\frac{1}{2}\int_\Omega X^2(\mathbf{x}) + x^2 + Y^2(\mathbf{x}) + y^2 d\lambda_3(\mathbf{x}) - f^2\int_\Omega \mathbf{x}.\mathbf{X}(\mathbf{x})d\lambda_3(\mathbf{x}), \end{aligned}$$

where λ_3 denotes volume (or more precisely 3-dimensional Lebesgue measure). Suppose one possible state of the fluid at a fixed time t is described by a function $\mathbf{X}_t = (X_t, Y_t, Z_t)$. (One way to calculate a candidate state of the fluid would be to solve equation (51) on particles, i.e. solve (53) for $\tilde{\mathbf{X}}(t,.)$, and then assume the particles at time t are in the same configuration that they are in at time 0.) Values are fixed on particles, but the particles may take up any configuration (subject to the incompressibility constraint that they do not change volume), therefore possible states of the fluid are described by elements of $\mathcal{R}(\mathbf{X}_t)$. The Cullen-Norbury-Purser principle states that we should minimise the geostrophic energy over all possible states, therefore we study the minimisation problem

$$\inf_{\mathbf{X}\in\mathcal{R}(\mathbf{X}_t)} E(\mathbf{X}), \tag{55}$$

where the energy minimiser (if it exists and is unique) gives the actual state of the fluid. The following theorem is due to Douglas (1998, Theorem 3.1) (in which we make an additional physically reasonable assumption which ensures (55) is finite):

Theorem 8 *Suppose that $\mathbf{X}_t \in L^p(\Omega, \lambda_3, \mathbb{R}^3)$, where $2 \le p < \infty$. Then (55) is uniquely attained by the monotone rearrangement of $\mathbf{X}_t$, that is the unique element of $\mathcal{R}(\mathbf{X}_t)$ equal to the gradient of a convex function.*

Proof. Applying Theorem 4 yields that $\int_\Omega X^2(\mathbf{x}) + Y^2(\mathbf{x})d\lambda_3(\mathbf{x})$ has the same value for each $\mathbf{X} \in \mathcal{R}(\mathbf{X}_t)$: therefore it is sufficient to study

$$\sup_{\mathbf{X}\in\mathcal{R}(\mathbf{X}_t)} \int_\Omega \mathbf{x}.\mathbf{X}(\mathbf{x})d\lambda_3(\mathbf{x}). \tag{56}$$

Theorem 6 yields that (56) is uniquely attained by $\mathbf{X}_t^\#$, the monotone rearrangement of $\mathbf{X}_t$. □

We note that in simple examples equation (51) may be easy to solve analytically, and in such cases the methods of Cullen, Norbury and Purser described above have been used

to calculate solutions to the semigeostrophic equations. In general, however, this is not the case: to prove the existence of solutions, the equations are reformulated in $\mathbf{X}$ (dual) space as a coupled Monge-Ampère/ transport problem. We describe how this is done in §6.5 and discuss existence results in this setting in §6.6.

Benamou (1992) interprets the minimum energy principle in the following way: assume that the configuration of the fluid at each time t is an energy minimum with respect to permutation of the fluid particles. Recalling from (50) that $\mathbf{X} = \nabla P$, and using the methods of Brenier (1991, Proposition 2.1), it can be shown that $P(t,.)$ must be convex. The approach used here differs in that we prove that an energy minimiser exists (and is unique), rather than assuming it exists.

6.3 The semigeostrophic equations as a model for frontogenesis

The energy minimisation problem of the previous subsection yielded that stable solutions of the semigeostrophic equations satisfy $\mathbf{X}(t,.) = \nabla P(t,.)$ at each time t for some convex function $P(t,.)$. In this subsection, following (for example) Cullen (1983) and Evans (1997, Section 7.3), we interpret singularities of $P(t,.)$ as weather fronts, that is regions across which there are large variations in wind and temperature. As time evolves, we track the singularities of $P(t,.)$; this is thought of as weather fronts forming and moving. This is the sense in which the semigeostrophic equations are considered a model for frontogenesis. In this subsection we merely outline this idea, giving references to some relevant literature.

To study the regularity of $P(t,.)$, we note that $\mathbf{X}(t,.)$ is the unique minimiser of

$$\inf_{\mathbf{X} \in \mathcal{R}(\mathbf{X}_t)} \int_\Omega |\mathbf{X}(\mathbf{x}) - \mathbf{x}|_2^2 d\lambda_3(\mathbf{x}),$$

and as described in §5 this may be rewritten as an optimal mass transfer problem, which has an associated regularity theory. (See §5.4.) For each time t, define a (Borel) measure ν_t by $\nu_t(B) = \lambda_3(\mathbf{X}(t,.))^{-1}(B))$ for Borel subsets B of $\mathbb{R}^3$. The image space of $\mathbf{X}$ is known as *dual space*; ν_t measures what volume of physical space is mapped to a given set in dual space. We noted in §5 that the set of measure-preserving mappings between (Ω, λ_3) and $(\mathbb{R}^3, \nu_t)$ is exactly $\mathcal{R}(\mathbf{X}(t,.))$. Our regularity theory holds in the special case when ν_t satisfies some additional properties. For a given t, suppose that $\mathbf{X}(t,.)$ is nondegenerate in the sense of Brenier (1991), that is $\mathbf{X}(t,.)$ does not map a set of positive volume to a set of zero volume. Then $\nu_t(E)$ is well defined for Lebesgue measurable subsets of $\mathbb{R}^3$, and moreover ν_t is absolutely continuous with respect to Lebesgue measure; the crucial consequence of this is that we may write ν_t as

$$\nu_t(E) = \int_E \rho(t, \mathbf{X}) d\lambda_3(\mathbf{X})$$

for Lebesgue measurable subsets $E \subset \mathbb{R}^3$, where $\rho(t,.)$ is a non-negative integrable function which we call the *pseudo-density*. (In the language of measure theory, $\rho(t,.)$ is the Radon-Nikodym derivative of ν_t with respect to Lebesgue measure.) Define the *support* of $\rho(t,.)$, denoted supp $\rho(t,.)$, to be the smallest closed subset $F \subset \mathbb{R}^3$ such that $\lambda_3(\Omega) = \nu_t(F)$. To study the singularities of $P(t,.)$, we consider the regularity of $\nabla P(t,.)$, the optimal mass transfer from (Ω, λ_3) to (supp $\rho(t,.), \nu_t$) with respect to the cost $c(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2/2$.

Recall from §5 that regularity of $\nabla P(t,.)$ cannot be guaranteed in the case where supp $\rho(t,.)$ is not convex. Therefore loss of convexity of supp $\rho(t,.)$ as time evolves can be seen as a mechanism for introducing singularities of $P(t,.)$. A computationally appealing example is the following. Suppose Ω is convex, and let $\rho(0,.)$ be a convex pseudo-density

patch, that is 1 on some bounded convex set and 0 otherwise; now let time evolve. If the geostrophic velocity is sufficiently regular, we see in §6.5 that $\rho(t,.) \in \mathcal{R}(\rho(0,.))$ for each t, therefore $\rho(t,.)$ will continue to be the characteristic function of some set (i.e. 1 on some set, 0 elsewhere). However supp $\rho(t,.)$ may lose convexity as time evolves, and singularities of the optimal mapping may form and move. The author is not aware of any such numerical computations.

The 2-dimensional semigeostrophic equations, that is (45), and the appropriate forms of (47) and (48), are still a physically relevant system in certain regimes. They can be studied in exactly analogous fashion to the 3-dimensional equations. As explained in §5, additional regularity results for the optimal mapping are available when we restrict to two dimensions. In particular we can interpret Wolfson's result (1997) on the existence of an area-preserving diffeomorphism as showing that fronts do not form when the pseudo-density is a characteristic function, and Ω and supp $\rho(t,.)$ satisfy a "joint convexity condition". (See §5 for details, but note that if Ω is a ball, and supp $\rho(t,.)$ is "nearly convex", then fronts will not form.) Computations have been performed for two-dimensional semigeostrophic models: Cullen and Roulstone (1993) numerically simulated the evolution of an unstable wave as it passes through frontal collapse.

As stated in Evans (1997, Section 7.3), "There are extremely interesting mathematical problems here, which have only in part been studied."

6.4 Identification of the trajectory mapping via polar factorisation

At each fixed time t, stable solutions (in the sense of §6.2) of the semigeostrophic equations satisfy $\mathbf{X}(t,.) = \nabla\psi(t,.)$ for some convex function $\psi(t,.)$ which we identify with $\phi/f^2 + (x^2+y^2)/2$. In this subsection we study the Lagrangian version of the problem, that is equations (52), (53) and (54), using the techniques of Benamou and Brenier (1992, 1998a). Our aim is to find the trajectory mapping χ.

For each t, assume $\tilde{\mathbf{X}}(t,.)$ is square integrable and nondegenerate (as defined in §5.5, that is $\tilde{\mathbf{X}}(t,.)$ does not map a set of positive size to a set of zero size). Then, for each t, Theorem 7 yields that $\tilde{\mathbf{X}}(t,.)$ has a unique polar factorisation into the composition of its monotone rearrangement with a measure-preserving mapping, that is there exists a convex function $\psi(t,.)$, and a unique measure-preserving mapping $s(t,.)$ such that

$$\tilde{\mathbf{X}}(t,.) = \nabla\psi(t, s(t,.)). \tag{57}$$

We identify $\psi(t,.)$ with $P(t,.)$ (as before), and $s(t,.)$ with the trajectory mapping $\chi(t,.)$ at each fixed time t. For each t, Proposition 4 yields that $\chi(t,.)$ is the projection of $\tilde{\mathbf{X}}(t,.)$ onto the set of measure-preserving mappings from Ω to Ω. Denoting this projection mapping by Π_S, (53) yields that

$$\frac{\partial\tilde{\mathbf{X}}}{\partial t} = f(J(Id - \Pi_S)\tilde{\mathbf{X}}) \tag{58}$$

where Id denotes the identity map on Ω. Benamou (1992) refers to (58) as the *dynamic rearrangement equation.*

The nondegeneracy assumption implies that $\tilde{\mathbf{X}}(t,.)$ has no level sets of positive measure: this excludes some physically reasonable solutions. As discussed in §5.5, if $\tilde{\mathbf{X}}(t,.)$ has at least one level set of positive measure, then the polar factorisation (57) (if such exists) is not unique: hence in this case we cannot define the trajectory mapping uniquely without further information. A question of current interest is how to choose the trajectory mapping when the polar factorisation (57) is known to exist but is not unique.

6.5 Reformulation of the semigeostrophic equations as a coupled Monge-Ampère/transport problem

In §6.2 we showed that solutions of the semigeostrophic equations (50), (51) and (47) which can be interpreted as a sequence of minimum energy states are those for which $P(t,.)$ is a convex function at each time t. However we did not address the problem of showing that such solutions exist. Our aim in this subsection is to rewrite the semigeostrophic equations as a coupled Monge-Ampère/ transport problem in $\mathbf{X}$ (or dual) space; we discuss existence of solution for this formulation in the next subsection. We assume that $\mathbf{X}(t,.)$ is nondegenerate for each t, and show that the (Legendre–Fenchel) conjugate convex function of $P(t,.)$ (which we denote $R(t,.)$) satisfies a Monge-Ampère problem in a generalised sense. The right-hand side of the Monge-Ampère equation is the pseudo-density (at time t). The Monge-Ampère problem replaces (50), the convexity condition and the nondegeneracy assumption. (This reformulation uses the so-called duality structure of the semigeostrophic equations.) We can use (51) to derive an evolution equation for the pseudo-density; this yields a coupled system which is a closed form of the semigeostrophic equations. The advecting velocity is the geostrophic velocity (written in dual space coordinates); it may be derived from R.

To establish the coupled problem in dual space and to discuss existence results we need to use technical language: the reader is referred to Friedman (1982) for a more detailed description of the terms used. We first address the question of why we reformulate the semigeostrophic equations in dual space, rather than establish an existence theory in the physical space Ω. To use the results of Diperna and Lions (1989) to show existence of solution for the evolution equation (51) we require the advecting velocity $\mathbf{u}(t,.) \in W^{1,1}(\Omega)$ (or better) at each time t, that is $\mathbf{u}(t,.)$ is integrable, and has a derivative in the sense of integration by parts against smooth functions which vanish on $\partial\Omega$, which is also integrable. However the convexity of $P(t,.)$ at each time t gives information about the regularity of $\mathbf{u}_g$ (via (50)) rather than $\mathbf{u}$. To avoid this difficulty we work in dual space.

The methods of §6.2 yielded $\mathbf{X}$ as a function of $\mathbf{x}$ at each fixed time t; now we wish to reformulate the semigeostrophic equations in $\mathbf{X}$ variables, therefore we find a mapping from $\mathbf{X}$ to $\mathbf{x}$. For each t, given $\mathbf{X}(t,.) = \nabla P(t,.)$, essentially we restrict attention to the case where $\nabla P(t,.)$ is injective, so that we have $\mathbf{x}(t,.) = (\nabla P(t,.))^{-1}$. For a convex function there is a natural notion of generalised derivative, the subdifferential: the *subdifferential of f at a point* $\mathbf{x}$, which we denote $\partial f(\mathbf{x})$, is the set of hyperplanes to the graph of f at the point $(\mathbf{x}, f(\mathbf{x}))$. A convex function may have "kinks", but it is differentiable at all other points on the interior of the set where it is finite: the subdifferential of a convex function f is the singleton set $\{\nabla f(\mathbf{x})\}$ for almost every $\mathbf{x}$ where $f(\mathbf{x})$ is finite. Without loss of generality we can choose $P(t,.)$ to be lower semicontinuous (which, roughly speaking, is a one-sided version of continuity), and proper, that is $P(t,.)$ never takes the value $-\infty$ and is finite at some point. We define the (Legendre-Fenchel) *conjugate convex* function of $P(t,.)$, which we denote $R(t,.)$, by

$$R(t,\mathbf{X}) = \sup\{\mathbf{x}.\mathbf{X} - P(t,\mathbf{x}) : \mathbf{x} \in I\!R^3\}. \tag{59}$$

We have the following result.

Proposition 5 *Suppose $\mathbf{X}(t,.) : \Omega \to I\!R^3$ is an integrable function which satisfies $\mathbf{X}(t,.) = \nabla P(t,.)$ almost everywhere for some proper lower semicontinuous convex function $P(t,.) : I\!R^3 \to I\!R \cup \{+\infty\}$. Then, for almost every $\mathbf{x} \in \Omega$,*

$$P(t,\mathbf{x}) + R(t,\mathbf{X}(t,\mathbf{x})) = \mathbf{x}.\mathbf{X}(\mathbf{x},t). \tag{60}$$

Moreover if $\mathbf{X}(t,.)$ *does not map a set of positive size to a set of zero size (i.e.* $\mathbf{X}(t,.)$ *is nondegenerate), then*

$$\mathbf{x} = \nabla R(t, \mathbf{X}(\mathbf{x}, t)) \tag{61}$$

for almost every $\mathbf{x} \in \Omega$.

Proof. From our remarks above we have that $\partial P(t,.)(\mathbf{x}) = \{\mathbf{X}(t,\mathbf{x})\}$ for almost every $\mathbf{x} \in \Omega$. Standard convex analysis (see Rockafellar 1970, Theorem 23.5) demonstrates that (60) holds for almost every $\mathbf{x} \in \Omega$. Noting that $P(t,.)$ is convex, lower semicontinuous and proper, and adopting the notation that f^* denotes the conjugate convex function of f, Rockafellar (1970, Theorem 12.2) yields that $P^{**}(t,.) = P(t,.)$. Rewriting (60) we have

$$R^*(t,\mathbf{x}) + R(t, \mathbf{X}(t,\mathbf{x})) = \mathbf{x}.\mathbf{X}(t,\mathbf{x})$$

for a.e. $\mathbf{x} \in \Omega$. It follows that $\mathbf{x} \in \partial R(t,.)(\mathbf{X}(t,\mathbf{x}))$ for almost every $x \in \Omega$. If $\mathbf{X}(t,.)$ is nondegenerate, then $\partial R(t,.)(\mathbf{X}(t,\mathbf{x})) = \{\nabla R(t, \mathbf{X}(t,\mathbf{x}))\}$ for a.e. $\mathbf{x} \in \Omega$, and we obtain (61). □

The nondegeneracy assumption ensures that $\mathbf{X}(t,.)$ does not map a set of positive size to the set where $R(t,.)$ is not differentiable, therefore (61) holds. For two points $\mathbf{x}, \mathbf{y} \in \Omega$, $\mathbf{x} \neq \mathbf{y}$, with $\mathbf{X}(t,\mathbf{x}) = \mathbf{X}(t,\mathbf{y}) = \mathbf{k}$, say, the methods of the above proof yield that $\mathbf{x}, \mathbf{y} \in \partial R(t,.)(\mathbf{k})$, therefore $R(t,.)$ is not differentiable at $\mathbf{k}$. So if we cannot make $\nabla P(t,.)$ injective by ignoring a set of measure zero, (61) is replaced by a differential inclusion. We do not study this problem in this article; however we note that it is the appropriate generalisation to the case where we do not assume the (physically unjustified) nondegeneracy condition.

At each fixed t, if we restrict $\mathbf{X}(t,.)$ to $\Omega_t \subset \Omega$, where Ω_t is chosen so that (61) holds for every $\mathbf{x} \in \Omega_t$ and $\lambda_3(\Omega_t) = \lambda_3(\Omega)$, then $\mathbf{X}(t,.)$ is injective on Ω_t. Now we can define the inverse mapping $\mathbf{x} = \mathbf{x}(t,\mathbf{X})$, and $\mathbf{x}(t,.) = \nabla R(t,.)$ on $\mathbf{X}(\Omega_t)$. $\mathbf{X}(t,.) = \nabla P(t,.)$, (60), and $\mathbf{x}(t,.) = \nabla R(t,.)$ are sometimes referred to as the *duality structure* of the semigeostrophic equations. There is a considerable literature which has identified and exploited this structure for particular solutions (see for example Cullen and Purser 1984, Chynoweth, Porter and Sewell 1988).

For each t let ν_t be the measure introduced in §6.3. $\mathbf{X}(t,.)$ is nondegenerate by assumption, therefore (as discussed in §6.3) ν_t may be written

$$\nu_t(E) = \int_E \rho(t,\mathbf{X}) d\lambda_3(\mathbf{X})$$

for every (Lebesgue measurable) subset E of $\mathbb{R}^3$, where (integrable non-negative) $\rho(t,.)$ is the pseudo-density. In particular at each time t we have the constraint

$$\int_{\mathbb{R}^3} \rho(t,\mathbf{X}) d\lambda_3(\mathbf{X}) = \int_{\mathbb{R}^3} d\nu_t = \lambda_3(\Omega).$$

Moreover the methods of Burton and Douglas (1998, Lemma 6) yield that $\mathbf{x}(t,\mathbf{X}) = \nabla R(t,\mathbf{X})$ for ν_t a.e. $\mathbf{X} \in \mathbb{R}^3$, or equivalently, for a.e. $\mathbf{X} \in \text{supp } \rho(t,.)$ (where supp $\rho(t,.)$ is the smallest closed set which contains all the mass of ν_t).

We rewrite (50) using the methods of Brenier (1991). Fixing t, for each $F \in C_c(\mathbb{R}^3)$ (where $C_c(\mathbb{R}^3)$ denotes continuous functions with compact support in $\mathbb{R}^3$) it follows from (50) that

$$\int_\Omega F(\nabla P(t,\mathbf{x})) d\lambda_3(\mathbf{x}) = \int_{\mathbb{R}^3} F(\mathbf{X}) \rho(t,\mathbf{X}) d\lambda_3(\mathbf{X}). \tag{62}$$

If we make the substitution $\mathbf{X} = \nabla P(t, \mathbf{x})$ in the left hand integral of (62), a formal calculation, using the fact that $\mathbf{x}(t, \mathbf{X}) = \nabla R(t, \mathbf{X})$ on supp $\rho(t, .)$, yields

$$\int_{\text{supp } \rho(t,.)} F(\mathbf{X}) \det H(R(t, \mathbf{X})) d\lambda_3(\mathbf{X}) = \int_{\mathbb{R}^3} F(\mathbf{X}) \rho(t, \mathbf{X}) d\lambda_3(\mathbf{X})$$

for each $F \in C_c(\mathbb{R}^3)$, where H denotes the Hessian matrix. This gives the following Monge-Ampère problem for each t:

$$\det H(R(t, \mathbf{X})) = \rho(t, \mathbf{X}), \tag{63}$$
$$\nabla R(t, \mathbf{X}) \text{ maps the support of } \rho(t, \mathbf{X}) \text{ into } \overline{\Omega}.$$

It follows that (62) is a weak formulation of (63).

We can use the evolution equation (51) to derive the following transport equation for ρ, which holds in the classical weak sense (see Benamou and Brenier 1992, 1998a for details):

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{U}) = 0 \tag{64}$$

where $\mathbf{U} = f(y - Y, X - x, 0)$ and $\nabla \equiv (\partial/\partial X, \partial/\partial Y, \partial/\partial Z)$. Noting that $\mathbf{x}(t, .) = \nabla R(t, .)$ ν_t a.e. in $\mathbb{R}^3$ for each t, we have

$$\mathbf{U} = f(J(Id - \nabla R)) \tag{65}$$

where Id denotes the identity mapping on $\mathbb{R}^3$. The transport equation (64) is coupled to the Monge-Ampère problem (62) by (59) and (65). It follows from (65) that if R is sufficiently smooth, $\nabla.\mathbf{U} = 0$. When this holds, the solution of (64) can be viewed as rearranging the initial values of ρ i.e. $\rho(t, .)$ is a rearrangement of $\rho(0, .)$ for each positive time t.

6.6 Weak solutions of the semigeostrophic equations

The previous section yielded equations (59), (62), (64) and (65) for unknowns R, P, ρ and $\mathbf{U}$. We discuss existence of solution for the system (62) and (64) (which holds in the classical weak sense), coupled by (59) and (65), posed on $[0, t_1) \times \mathbb{R}^3$ for some $t_1 > 0$, with initial condition $\rho(0, .) = \rho_0$ given. The difficulty is showing that $\mathbf{U}$ is sufficiently regular to solve the evolution equation (64) in some suitable sense: in view of (65), what we have to show is that given $\rho(t, .)$, the function $R(t, .)$ obtained from (62) and (59) is sufficiently smooth. One possible strategy to prove existence of solutions is to use a fixed point argument. From our earlier assumptions we have that $\rho(t, .)$ is non-negative and integrable; if we could show that the solution to the Monge-Ampère problem $R(t, .) \in W^{2,1}(\mathbb{R}^3)$, then (65) yields that $\mathbf{U}(t, .) \in W^{1,1}(\mathbb{R}^3)$, and we could use the transport theory of Diperna and Lions (1989) to solve (64). The author is not aware of any proof of this result. Given the additional hypothesis that $\rho_0 \in L^p(\mathbb{R}^3)$ for $p > 3$, and is compactly supported, Benamou and Brenier (1992, 1998a) proved a result of weak existence - they showed existence of solutions for an approximate problem and then passed to the limit. The smoother approximate problem was obtained by discretising time and mollifying; the fact that a limiting weak solution exists is reliant on the continuity of the mapping $u \to u^{\#}$ (where $u^{\#}$ denotes the monotone rearrangement of u). Similar results were obtained by Otto (1997). The author is not aware of any uniqueness results.

For a given t, let S_t denote the set of measure-preserving mappings from (supp $\rho(t, .), \nu_t) \to (\Omega, \lambda_3)$; it is easily seen that $\nabla R(t, .)$ is the optimal mapping in S_t against the cost $c(W, Z) = |W - Z|^2/2$. We can choose the "target set" Ω convex, so can we use the

regularity theory for $R(t,.)$ stated in §5.4 to prove existence of strong solutions (at least for short times)? This question is open to the best of the author's knowledge. For notational convenience we write $Y(t)$ for the interior of supp $\rho(t,.)$. If we assume that Ω and supp $\rho(t,.)$ are bounded, convex and have smooth boundaries, Caffarelli (1996) shows that if $\rho(t,.) \in C^\alpha(Y(t))$ (for some $0 < \alpha < 1$), and is bounded above and below, away from zero, then $R(t,.) \in C^{2,\alpha}(\overline{Y(t)})$: in this case we have regularity up to the boundary, and classical transport theory (using characteristics) may be applied. However the same regularity theory yields $P(t,.) \in C^{2,\alpha}(\overline{\Omega})$; in the light of §6.3, restricting to solutions where the support of $\rho(t,.)$ is convex and has smooth boundary is limiting the theory to the case when no fronts form! If we relax the assumptions to Ω convex, supp $\rho(t,.)$ bounded and connected, (with the same restrictions on $\rho(t,.)$,) then there is an interior regularity theory due to Caffarelli (1992a). If $\rho(t,.) \in C^{1,\beta}(Y(t))$ for some $0 < \beta < 1$, then $R(t,.) \in C^{3,\alpha}_{loc}(Y(t))$ for each $0 < \alpha < \beta$. The challenge is to demonstrate regularity up to the boundary (e.g. $R(t,.) \in C^{2,\alpha}(\overline{Y(t)})$) in this case.

We conclude this section with an observation of Evans (1997). The system (64), (65) and (63), posed on $[0,t_1) \times Y(t)$, is similar in structure to the vorticity formulation of the 2-dimensional Euler equations:

$$\frac{\partial \omega}{\partial t} + \nabla.(\omega \mathbf{u}) = 0, \tag{66}$$

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right), \tag{67}$$

$$-\Delta \psi = \omega. \tag{68}$$

Here $\mathbf{u}$ is the velocity, ω the vorticity, ψ the streamfunction. For the semigeostrophic equations, we have a Monge-Ampère problem rather than (68): roughly speaking, (66), (67) and (68) is a linearisation of (64), (65) and (63). There is an existence theory (for strong global solutions) for the 2-dimensional Euler equations, (see for example Lions 1996,) so we can hope that a parallel theory can be developed for the semigeostrophic equations.

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